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Reflections on the Metaphysics of the Infinitesimal Calculus

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I seek to know in what consists the true sense of the infinitesimal Calculus; the reflections that I have laid forward on this subject are set in three chapters: in the first I state the general principles of infinitesimal analysis; in the second I examine how this analysis was reduced to algorithms by the invention of the differential and integral Calculus; in the third I compare this analysis to other methods which can replace it, such as the method of exhaustion, that of indeterminates and indivisibles, that of indeterminates, etc.

First Chapter

General Principals of Infinitesimal Analysis

1. There is no discovery in the mathematical sciences that has produced so fortunate and prompt a revolution as that of infinitesimal analysis; none has furnished either simpler or more efficacious means to penetrate into the knowledge of the laws of nature. In decomposing, so to speak, bodies into their elements, they seem to have indicated their interior structure and organization; but, since all that is extreme escapes from the senses and the imagination, one has only ever been able to form an imperfect idea of these elements, species of singular beings, which sometimes play the role of real quantities, and sometimes must be treated as absolutely nothing and seeming, by their equivocal properties, to take the middle ground between quantity and zero, between being and nothingness.[*]

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[*]§1 – I speak here in conformance with the vague ideas held regarding the quantities commonly called *infinitesimals,* as one has not taken the pain to examine their nature; but, in truth, nothing is simpler than the exact notion of these sorts of quantities. In effect, to say of a quantity that it is infinitely small, is precisely to say that it is the difference between two magnitudes which have the same third magnitude as their limit, and nothing more than this. The idea of an infinitesimal quantity is thus not more difficult to grasp than that of a limit; but it has, as everyone agrees, the advantage of leading to a much simpler theory.[*]§13 – I suppose here that the proposed question had beforehand been reduced to finding, in effect, the relationships which exist between such or such proposed quantities. If, for example, we are concerned with drawing a tangent to any underdetermined point of this curve, I begin by arbitrarily fixing the point through which I wish to draw the tangent, and I reduce the question to finding the relationship which exists, for example, between the sub-tangent and the abscissa, or between the ordinate and the sub-normal corresponding to the same point. But if someone asked me, for example, how I would apply my definition of the *infinite* which we shall see, to these questions: *Is matter infinitely divisible? Is the space in which exist all created things infinite?* and other similar questions; I respond that my definition is only that of the mathematical infinite; that it can only be applied to questions of which the object is only to find relationships which exist between

Luckily, this difficulty has not injured the progress of the discovery: it is certain primitive ideas which always leave some clouds in the mind, but of which the first consequences, once drawn out, open a vast field, easy to traverse. Such had seemed that of the infinite, and several geometers made the most happy use of it, who could not perhaps have fathomed the notion at all; however the philosophers could not content themselves with an idea so vague: they wanted to ascend to the principles, but they found themselves divided in their opinions, or more so in their manner of envisioning the objects. My goal in this writing is to bring together these different points of view, to show the relationships between them, and to propose new ones. I will think myself well rewarded by my work if I am able to succeed in throwing some light on such an interesting subject.

2. The difficulty that one often encounters, in explaining exactly through equations the different conditions of a problem, and in resolving these equations, could have given birth to the first ideas of the infinitesimal Calculus. When it is too difficult, as a matter of fact, to find the exact solution to a question, it is natural to look for a means of approaching it as closely as possible, in leaving out those quantities which hinder the combinations, if one foresees that these neglected quantities can, because of their smallness of value, produce only a slight error in the results of the calculations. It is thus, for example, that, only being able to discover with difficulty the properties of curves, one could have imagined them as polygons with a large number of sides. As a matter of fact, if e.g., one conceives of a regular polygon inscribed in a circle, it is apparent that these two figures, though always different and never capable of becoming identical, do, however, come to resemble each other more and more, accordingly as the number of sides of the polygon increases, that their perimeters, their areas, the solids formed by their revolutions around a given axis, analogous lines drawn inside or outside these figures, the angles formed by these lines, etc., are, if not respectively equal, are at least approaching equality to the degree that the number of sides becomes greater, from which it follows that, in supposing the number of sides to be very large in fact, one could without sensible error attribute to the circumscribed circle the same properties pertain to the inscribed polygon.

Furthermore, each of the sides of the polygon diminishes in size, obviously, accordingly as the number of sides increases, and as a consequence, if one supposes that the polygon be really composed of a very great number of sides, one could say also that each of them is really very small.

This posed, it a particular circumstance should be found by chance in the course of a calculation where one can much simplify the operations, in neglecting, for example, one of these small sides in comparison to a given line, such as the radius; in other words, it is clear that we could, without any disadvantage, use in our equations the given line instead of a quantity equal to the sum of this line and one of these small sides, for the error which would result could only be extremely small and would not merit putting ourselves through the trouble to come to know its value.

3. For example, let us draw a tangent at a given point M on the circumference MBD (*Fig. 1*).

Let C be the center of the circle, DCB the axis; let us suppose the abscissa DP = x, the corresponding ordinate MP = y, and let TP be the sought sub-tangent.

To find it, let us consider the circle as a polygon with a large number of sides. Let MN be one of these sides; and let us prolong it up to the axis: this will obviously be the tangent in question, since this line will not penetrate into the interior of the polygon. Drop also the perpendicular MO on NQ, parallel to MP, and name the radius of the circle a; this posed, we will clearly have

MO : NO : : TP : MP, or MO/NO = TP / y.

In addition, the equation of the curve being, for the point M, $yy = 2ax - xx$, it will be, for the point N,

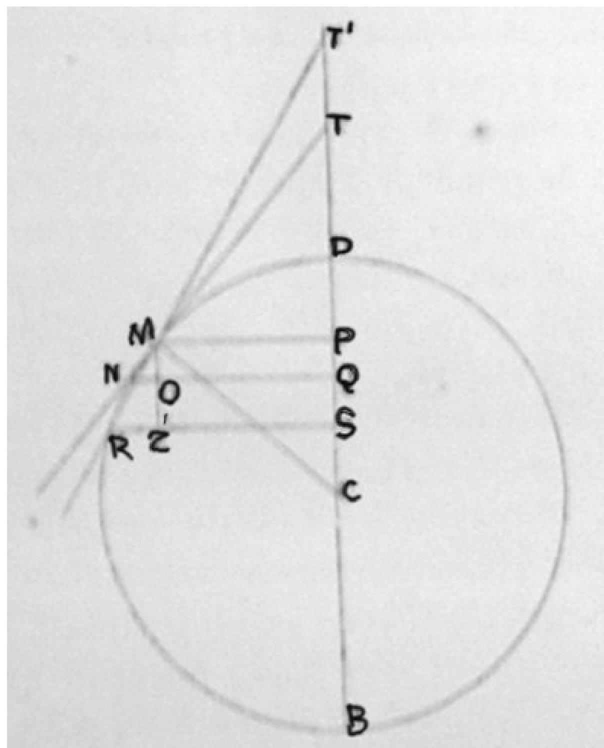
$(y + \text{NO})^2 = 2a\ (x + \text{MO}) - (x + \text{MO})^2$;

subtracting the first equation (found for point M) from this equation (found for point N), and then reducing, we have

$$\frac{MO}{NO} = \frac{2y + NO}{2a - 2x - MO};$$

setting equal therefore this value of MO/NO to that found above, and multiplying by y, there follows

$$TP = \frac{y(2y + NO)}{2a - 2x - MO}.$$



If therefore MO and NO are known, we will have the sought value of TP; now these quantities MO, NO are very small, since there are each less than the side MN, which, by hypothesis, is itself very small. Therefore (by section 2) one can neglect without any sensible error these quantities by comparison with the quantities 2y and 2a – 2x to which they are added. Therefore the equation is reduced to

$$TP = \frac{y^2}{a - x},$$

which was to be found. Q.E.D.

4. If this result is not absolutely exact, it is at least evident that in practice it can pass for such, since the quantities MO, NO are extremely small; but someone who had no idea of the doctrine of infinites would be perhaps much amazed if we tell him that the

equation $TP = \frac{y^2}{a-x}$ not only approaches the truth, but really is of the most perfect exactitude: it is however a thing easy to be certain of in finding TP, using the principle that the tangent is perpendicular to the extremity of the radius, for it is visible that the similar triangles CPM, MPT give CP : MP :: MP : TP, from which one derives $TP = \frac{(MP)^2}{CP} = \frac{y^2}{a-x}$, as above.

5. For a second example, let us suppose the question of finding the area of a given circle.

Let us again consider this curve as a regular polygon with a large number of sides; the area of any regular polygon is equal to the product of its perimeter and half the perpendicular drawn from the center to an edge; therefore, the circle being considered as a polygon of many sides, its area must be equal to the product of its circumference by half its radius: a proposition that is not less exact than the result found above.

6. However vague and of little precision may seem these two expressions *very great* and *very small,* or other equivalents, one sees, by the two preceding examples, that it is not without utility that one employs them in mathematical combinations and that their usage can be of a great help in facilitating the solution of diverse questions that can be proposed, for, once their notion is admitted, all curves can, as well as the circle, be considered as polygons of many sides; all surfaces can be divided into a multitude of bands or zones, all bodies into corpuscles; all quantities, in a word, can be decomposed into particles of the same sort as them [the quantities]. From this spring many new relationships and new combinations, and one can easily judge, by the examples cited above, the resources that must be furnished to calculation by the introduction of these elementary quantities.

7. But that advantage that they procure is yet well more considerable than we first had room to hope it to be, for it follows from the reported examples that what was seen at first only as a simple method of approximation, leads us, at least in certain cases, to perfectly exact results. It will therefore be interesting to know how to distinguish those where this arrives, to bring to this condition other cases as much as possible, and to change thus this method of approximation into a calculation perfectly exact and rigorous. Now, such is the object of infinitesimal analysis.

8. Let us see first how, in the equation found in section III, $TP = \frac{y(2y+NO)}{2a-2x-MO}$, it was possible to leave out MO and NO, without altering at all the correctness of the result; or rather how the result became exact by the omission of these quantities, and why it was not exact before their omission.

Now, one can give simple account for the reason for what has occurred in the solution of the problem treated above, in remarking that, the hypothesis from which we set out being false, since it is absolutely impossible that a circle can ever be considered as a real polygon, whatever may be the number of its sides; under this hypothesis there is necessarily a certain error in the equation $TP = \frac{y(2y + NO)}{2a - 2x - MO}$, and that the result $TP = \frac{y^2}{a - x}$ was nonetheless certainly exact, as we proved by the comparison of the two triangles CPM, MPT: we were able to leave out MO and NO in the first equation, and we indeed had to do so to rectify the calculation and eliminate the error which had been brought about by the false hypothesis from which we began. Leaving out the quantities of this nature is therefore not only permitted in such a case, but it is necessary, and it is the only manner of exactly explaining the conditions of the problem.

9. The exact result $TP = \frac{y^2}{a - x}$ was therefore only obtained by a compensation of errors, and this compensation can be rendered yet more apparent in treating the above reported example in a slightly different manner, that is by considering the circle as a real curve, and not as a polygon.

To this end, for a point R taken arbitrarily at some distance from point M, let there be drawn a line RS parallel to MP, and for the points R and M let there be drawn the chord RT'; we will obviously have

T'P : MP : : MZ : RZ

and therefore

T'P or (TP + T'T) = $MP\frac{MZ}{RZ}$.

This posed, if we imagine that RS moves parallel to itself in continually approaching MP, it is visible that the point T' will approach at the same time closer and closer to T and that one could consequently make the line TT' as small as we will like without the above-established proportion ceasing to hold. If therefore I leave out this quantity TT' in the equation that I just found, there will result in truth an error in the equation TP = MP (MZ / RZ), to which it will be reduced; but this error can be reduced as much as one would like, by causing RS to approach MP as much as is necessary, which is to say that the relationship of the two parts of this equation will differ from each other as little as one would like in respect to their equality.

Similarly, we have MZ / RZ = (2y + RZ) / (2a – 2x – MZ) (§3) and this equation is perfectly exact whatever be the position of point R, which is to say, whatever be the values of MZ to RZ. But the more RS approaches MP, the more the lines MZ and RZ will be small; and therefore if one leaves them out in the second part of that equation, the error which will result in the equation MZ / RZ = y / (a – x), to which it will then be reduced, can, as the first, be made a small as one would judge it for our concerns.

This being so, without having regard to the errors which I will always be master to reduce as much as I like, I treat the two equations that I just found,

TP = MP (MZ / RZ) and MZ/RZ = y/(a – x),

as if they were both perfectly exact; substituting therefore in the one the value of MZ/RZ derived from the other, I have as a result TP = y^2 / (a – x), as above.

This result is perfectly right, since it conforms to that which we obtained by comparison of the triangles CPM, MPT and even though the equations TP=y (MZ/RZ) and MZ/RZ=y/(a-x), from which it was derived, are certainly both false, since the distance from RS to MP was never proposed to be zero, nor even very small, but instead equal to a certain arbitrary line. It must consequently be necessary that the errors mutually compensate themselves in the comparison of the two erroneous equations.

10. We see, therefore, the occurrence of compensated errors well attained and proved; we now treat explaining it, to research the sign by which one recognizes that the compensation holds in calculations similar to the preceding, and the means to produce this compensation in each particular case.

Now, it suffices for this to remark that, the errors made in the equations TP = y MZ / RZ and MZ / RZ = y / (a – x) being able to be made as small as one likes, that error which will occur (if one is indeed to be found) in the resulting equation TP = y^2 / (a – x) can also be made as small as one would like, and it will depend on the arbitrary distance of the lines MP and RS. Now this is not so, since, once given the point M through which the tangent must pass, it is found that none of the quantities, *a*, *x*, *y*, or TP of this equation are arbitrary; therefore it cannot have, in effect, any error in this equation.

It follows from this that the compensation of errors, which were found in the equations TP = y MZ / RZ and MZ / RZ = y / (a – x), manifests itself in the result by the absence of the quantities MZ and RZ which caused the errors, and that consequently, after having introduced these quantities in the calculations to facilitate the expression of the conditions of the problem, and having treated them in the equations which expressed these conditions as null in comparison to the proposed quantities, [with the end of simplifying these equations], there is left only to eliminate these same quantities from the equations where they are still found to eliminate the errors that they have occasioned and to obtain a result which will be perfectly exact.

11. The inventor has therefore been able to be led to his discovery by a quite simple reasoning: if in the place of a proposed quantity, he was able to say,

"When I employ in the calculation another quantity which is not completely equal to it, there will result some error; but if the difference between the quantities employed is arbitrary, and if I am able to make it as small as I would like, this error will be by no means dangerous; I will even be able to commit many similar errors at the same time without any attendant disadvantage, since I will always remain master of the degree of precision that I would like to give to my results. There is still more: it is possible to make these errors mutually compensate and therefore my results become perfectly exact. But how operates this compensation, and how in every case?

"It is what a little reflection will be able to discover to us; in effect," the inventor will be able to say, "let us suppose for an instant that the desired compensation took place, and let us see by what sign it must manifest itself in the result of the calculation. Now, what must naturally happen, is that the quantities which occasioned these errors having disappeared, the errors have likewise disappeared, for, the quantities such as MZ and RZ, having by hypothesis been arbitrary values, they must no longer enter into the formulas or results that are not arbitrary, and that, being become exact by supposition, depend uniquely, not on the will of the calculator, but on the nature of the things of which one had proposed to find the relation expressed by these results. Therefore the sign that announces that the desired compensation occurs is the absence of the arbitrary quantities that produced these errors, and hence to establish this compensation, requires only the elimination of these arbitrary quantities."

12. To further cement these ideas and give to the principles which are thereby derived the degree of precision and generality that suits them, I will remark that the quantities that we have considered in the above-treated question can be distinguished into two classes; the first class is composed of quantities, which, like MC, MP, PT, and MT, are either given or determined by the conditions of the problem; and the second is composed of quantities, which, like RS, RT', and ST', depend on the arbitrary precision of point R, and such that as the position of this point R approaches the point M, each among them approaches its correspondent in the first class, so that MP, for example, is the limit of RS, which is to say, the fixed term which it continually approaches, or, if we like, its last value; likewise, MT is the limit or last value of RT', and PT that of ST'. For the same reason, it is clear that the limits or last values of MZ, RZ, MR, T'T are all 0; finally, it is also clear that the last ratio of RS to MP, that is, the last value of RS/MP is a ratio of unity, in the same way as that of RT' to MT, of ST' to PT, or, in short that of any other quantity to its limit.

13. Now let us now imagine then, to extend these remarks to other problems of the same genre, to any system of proposed quantities, and let it be a question of finding the relations which exist among them.[*]

14. First of all, I comprehend under the name of *designated quantities,* not only all the quantities which are proposed by the very statement of the question, but also all those which depend only on these quantities: that is, those which are functions of these and only these quantities.

15. I will call, on the contrary, *non-designated* or *auxiliary quantities* all those which are not at all part of the system of designated quantities, and which consequently do not enter essentially at all into calculations, but are introduced only to facilitate the comparison of proposed quantities.

Thus, in the preceding example, MP, MC, MT, DP, etc. are *designated* quantities, since they depend only on the position of point M through which the tangent must be drawn; but RS, and all those which depend on it, like MZ, RZ, T'T, T'P, etc. are *auxiliary* quantities, since we only imagined to draw them to aid the solution of the question, which was to find the relationship of MP to TP.

It clearly follows from this that in any non-designated quantity, there is necessarily something arbitrary; for, if in it there entered nothing arbitrary, its value would be assigned by the very conditions of the problem, and consequently would depend totally on the proposed quantities, which is contrary to the hypothesis.

16. When in mathematics, two lines, two surfaces, two solids, or any two quantities whatsoever are assumed to perpetually approach each other by insensible degrees, in a

manner that their ratio or quotient differs by less and less and differs as little as one likes from unity, one says that these two quantities have for their last ratio a ratio of equality.

17. If one of these magnitudes is an assigned quantity, and the other an auxiliary quantity, the first is called the *limit* or *least value,* of the second: that is to say, a *limit* is nothing other than a designated quantity to which an auxiliary quantity is assumed to approach perpetually, in such a manner that it is able to differ as little as one would like, and that their least ratio be a ratio of equality.

Also, it is only the auxiliary quantities, strictly speaking, that have what I call a limit; for the designated quantities not being assumed to change, but on the contrary being themselves the terms or last values of auxiliary quantities, cannot strictly be spoken of as having limits, unless we only say that every designated quantity is, itself, its own limit, one can only agree, since the least value of any determined quantity can only be the quantity itself.

18. Thus, in general we name least values and least ratios of quantities, the values of the ratios which are in effect the least of those that are assigned to these magnitudes and to their relations, by the law of continuity, when each of them is supposed to approach perpetually and by insensible degrees to the designated quantity to which they correspond.

19. One names in general an *infinitely small* quantity the difference between any auxiliary quantity and its limit; so, for example, RZ, which is the difference between RS and MP, is what one calls an infinitely small quantity.

20. We name on the contrary, *infinite,* or, *infinitely large,* every magnitude, which is equal to unity divided by an infinitely small quantity: such as, consequently, the quantity 1/RZ or 1/(RS-MP).

However, since the limit or least value of RS is MP, it is clear that the limit or least value of RZ or RS – MP is 0, and that of 1/RZ is 1 / 0.

21. So we could say in general that *an infinitely small magnitude is nothing other than a quantity whose limit is* 0*,* and that on the contrary, *an infinitely large quantity is none other than a quantity whose limit is* 1 / 0.

22. We comprehend under the name of *infinitesimal quantities,* infinite quantities or infinitely large quantities, and those which are infinitely small; all other magnitudes are called *finite quantities.*

23. To say, following vulgar usage, that the infinite is that without boundary, that without limit, or that of which the limit does not exist, is thus giving to it a simple idea that is not groundless, since in effect the infinitesimal quantities all have as limits, some 0, the others 1/0, which are not true quantities.

24. However from the limits of these quantities being 0 or 1/0, by no means does it follow that these quantities themselves are chimerical beings; for, on the contrary, by the

very definition (§19), an infinitely small quantity is the difference of two very real quantities, to wit: an ordinary auxiliary quantity and its limit.

25. It further follows from this that one can regard any infinitely small quantity as the difference between two auxiliary quantities which have as a limit a third similar designated quantity; for, let X and Y be two different auxiliary quantities which have as a limit a same third quantity A.

I say that X – Y is an infinitely small quantity. In effect, since the limit or last value of X is A, and that of Y is also A; it follows that the least value of X – Y will be A – A or 0. So the limit of A + (X – Y) is A; thus we can regard X – Y as the difference of an auxiliary quantity A + (X – Y) to its limit A; so (§19) this difference is an infinitely small quantity; thus we could say in general that *an infinitely small quantity is nothing other than the difference of two auxiliary quantities that have the same limit.*

26. Two quantities cannot have the same third quantity as a limit without having between each other, as last ratio, a ratio of equality; for, since by hypothesis, the limit or least value of X / A is 1, the same as that of Y / A; it is clear that the limit or least value of (X / A) / (Y / A) is also unity. Yet, (X / A) / (Y / A)= X / Y; thus the limit or last value of X / Y is 1, that is, that the last ratio of X to Y is a ratio of equality. So, in general, we could say that *an infinitely small quantity is the relationship of the difference between two magnitudes which have as last ratio, a ratio of equality to each of these magnitudes.*

27. Finally, it is evident that one could further say that *an infinitely small magnitude is none other than a non-designated quantity, to which one could attribute, at first, any arbitrary value, which one supposes next to decrease imperceptibly towards zero*. Thus, in general, when we say, *let Z, for example, be an infinitely small quantity*, that is precisely the same thing as if we were to say, *let Z be an ordinary arbitrary quantity* (and consequently auxiliary, for designated quantities cannot be arbitrary), *and let us suppose next that this quantity is decreasing perpetually towards zero.*

28. A quantity is called infinitely small, *relative* to another quantity, when the ratio of the first to the second is an infinitely small quantity, and reciprocally, the second is called infinite or infinitely great *relative to the first.*

29. Two quantities are deemed *to differ infinitely little,* or *to have infinitely little difference* from one another, when the ratio of one to the other differs from unity only by an infinitely small quantity, in a manner that their least ratio be a ratio of equality, RS and MP are plainly such quantities.

30. We name *infinitesimal calculus* the art which teaches to discover the ratios and any relations whatsoever which exist between the diverse parts of any system of proposed quantities, by aid of the quantities that I have just named infinitesimal.

These infinitesimal quantities, all being only auxiliary quantities, that is, introduced only in the calculation in order to facilitate the expression of the proposed conditions, it is clear that it is absolutely necessary to eliminate them from the calculation in order to obtain the desired result, that is, the sought ratios; thus we could say, in a way, that the infinitesimal calculus is an *unfinished* calculus, or an as-yet uncompleted calculus, because in effect as soon as one has arrived at eliminating from it the auxiliary quantities and those which do not enter essentially into the calculations, it ceases to be infinitesimal, and completely resembles ordinary algebraic calculus.[*]

To complete the explanation of the principal terms relevant to the theory of the infinite in general, there remains for me to say what I mean by *imperfect equation.*

31. I call an *imperfect equation* any equation of which the two members are unequal quantities, but differ infinitely little from one another, or, which comes to the same thing, every equation of which the two members, although unequal, have for least ratio, a ratio of equality.

So, for example, the false equations TP = y (MZ / RZ) and (MZ / RZ) = y/(a – x) found in (§9), are what I call imperfect equations, since the neglected quantities in the exact equations from which they are derived, are infinitely small quantities, thus it is on the theory of these sorts of equations which the solution of the above treated question and all similar types are founded. This is why I am going to explore the principles of this theory, which is the basis of infinitesimal calculus, or, rather, which is nothing other than the infinitesimal calculus itself.

FIRST THEOREM

32. *If in any imperfect equation, we substitute in place of any one of the quantities which enters therein, another quantity which differs infinitely little, or whose ratio to the first has unity for limit or last value, the equation which will result by this transformation could not be a false equation, that is it becomes absolutely exact, or at least it remains what I have called an imperfect equation.*In effect, since by hypothesis we have only substituted for one quantity another whose least value is the same, and whose ratio to the first has unity as limit, it is clear that this substitution can in no way change the least values of the members of the proposed equations, nor their least ratios. Yet this last ratio was, by hypothesis, unity before substitution; thus it will still be so after; so the equation will maintain the character of an imperfect equation, unless it becomes rigorously exact: which was to be proven.

SECOND THEOREM

33. *Every equation that contains only designated quantities cannot be an imperfect equation.*

In effect, by the definition of imperfect equations, their members are unequal; but differing infinitely little from one another, their ratio approaches as much as we want to a ratio of equality; thus some quantity enters in this equation which is not part of the system of proposed quantities; but by hypothesis, on the contrary, the here-proposed equation

[*]§30 – Everyone knows, in effect, that a calculation into which infinitesimal quantities enter is not considered completed, and that one relies on the exactitude of the result only at the moment where all these infinitesimal quantities are entirely eliminated.([†]) §36 – Coming soon: how this is derived.

contains only designated quantities. Thus it cannot be what I have called an imperfect equation: which was to be proven.

THIRD THEOREM

34. *Every imperfect equation to which we have performed only transformations similar to that indicated in the first theorem, and in which we shall succeed in eliminating by these transformations all non-designated quantities, will be necessarily and rigorously exact.*

For, by the first theorem it cannot be an absolutely false equation, and by the second, it cannot be an imperfect equation; thus it is necessarily and rigorously exact.

COROLLARY

35. Everything that was just said on the subject of imperfect equations must be understood to be equally true for proportions, propositions, and any arguments susceptible of being expressed by similar equations.

SCHOLIUM

36. Such are the general principles to which the theory of infinitesimal calculus is reduced. We see by these principles that, if having expressed by imperfect equations the conditions of the problem, we succeed then by transformations similar to that indicated in the first theorem, we succeed, I say, in eliminating from these equations all auxiliary or non-designated quantities, it will be necessary to, in the course of the calculations, perform a compensation of errors; and that the advantage of this calculation consists in the fact that the conditions of a question being often very difficult to express exactly by rigorous equations, while it would be easy to do it by imperfect equations, it gives the means to derive by these imperfect equations the same results and ratios with as much certitude as if the first equations had been truly of the most perfect exactitude, and this by the simple elimination of quantities whose presence occasions these errors.

The reason for this is simple: suppose that we have to discover the relations which exist among several proposed quantities; if it is difficult to directly find equations which express these relations, it is natural to resort to some intermediate quantities which serve as terms of comparison; by this means we could obtain, if not the sought equations, at least other equations where the proposed quantities would be mixed with these auxiliary quantities; thus it will no longer be a question of eliminating these. However, if moreover the values of these auxiliary quantities are arbitrary, and could be supposed as small as we want without there being any change to the proposed quantities, it is easy to perceive that if in the equations which express the sought relations, the arbitrary quantities are found mixed with the proposed quantities, each of these equations will be decomposable into two others, the one containing only designated quantities, and the other containing arbitrary quantities, more or less the same as an equation which contains real quantities and imaginary quantities could be decomposed in two, the one of real quantities, the other of imaginary quantities. Now, as we only need an equation which exists between proposed quantities, it is clear that we can, without any drawback, neglect the quantities which hinder the calculation in those equations which are mixed with arbitraries, as the errors which must therein result can lie only in an equation between the arbitraries that it contains. Yet this is precisely what happens in the infinitesimal calculus, as one treats as nulls, in comparison to finite quantities, those quantities which we have called infinitely small.

In order to render this explanation more sensible still, let us resume the above-treated example. We have found (§9),

$$TP{+}T'T = y\times\frac{MZ}{RZ}, \text{ and } \frac{MZ}{RZ}=\frac{2y+RZ}{2a-2x-MZ};$$

both perfectly exact equations, whatever be the values of MZ and RZ; so, taking from the first of these equations the value of MZ/RZ, and substituting it into the second, I have

$$\frac{TP+T'T}{y}=\frac{2y+RZ}{2a-2x-MZ},$$

an exact equation and which must therefore hold, whatever be the distance one would like to put between the lines RS and MP.

Now, it is easy to see(†) that I am able to put this equation in the following form:

$$\left(\frac{TP}{y}-\frac{y}{a-x}\right)+\left(\frac{T'T}{y}-\frac{yMZ+aRZ-xRZ}{(a-x)(2a-2x-MZ)}\right)=0,$$

in which the first term contains only given quantities or those determined by the conditions of the problem, and of which the second contains only arbitraries, and could be supposed as small as we would want without having any change to the quantities which are contained in the first term, since we are able to suppose RS as close as we want to MP. Thus, following the theory of indeterminates, each of the terms of this equation, taken separately, must be equal to zero; that is to say, that this equation could be decomposed into two others:

$$\frac{TP}{y}-\frac{y}{a-x}=0, \text{ and } \frac{T'T}{y}-\frac{yMZ+aRZ-xRZ}{(a-x)(2a-2x-MZ)}=0,$$

of which the first contains only designated quantities, and the second only arbitraries. But we only have need of the first, since it is this one which gives us the sought value of TP, as we have already found. So even though we would have committed errors in the course of the calculation, provided these errors only occur in the latter equation, the exactitude of the result would not have suffered at all; and this effectively is what would have happened if we had treated MZ, RZ and T'T as nulls by comparison to the proposed quantities a, x, y, in the original equations; we would have in truth committed errors in the expression of the conditions of the problem, but these errors would have destroyed themselves by compensation, and the needed result would have been in no way altered.

37. It is easy to perceive, according to what was just said, that infinitesimal analysis is nothing other than an application, or if you will, an extension of the method of indeterminates; for, following this method, I may say that when we neglect an infinitely small quantity, we do so, properly speaking, only to *infer,* and not to truly suppose it to be null; for example, when, instead of these two exact equations

$$TP+T'T=MP\times\frac{MZ}{RZ} \text{ and } \frac{MZ}{RZ}=\frac{2y+RZ}{2a-2x-MZ}$$

found in (§9), I employ the two imperfect equations

$$TP = MP \times \frac{MZ}{RZ}, \text{ and } \frac{MZ}{RZ} = \frac{y}{a-x};$$

I know quite well that I commit an error and I put them, so to speak, mentally under this form:

$$\frac{MZ}{RZ} \times MP = TP + \varphi, \text{ and } \frac{MZ}{RZ} = \frac{y}{a-x} + \varphi';$$

φ and φ' being such quantities as are necessary for the equations to hold exactly: likewise in the equation

$$\frac{TP}{MP} = \frac{y}{a-x},$$

resulting from the two imperfect equations above, I infer the quantity φ'' such that

$$\left(\frac{TP}{MP} - \frac{y}{a-x}\right) + \varphi'' = 0$$

be an exact equation; but I soon recognise that this last quantity φ'' is equal to zero, because if it were not zero, it could only be infinitely small, so long as there enters no other infinitesimal quantity in the first term; yet this is impossible, since each of these terms, taken separately, be not equal to zero; whence I conclude that we have exactly TP/MP = y/(a – x); and yet the quantities φ, φ', and φ'' have not been removed as nulls, but simply so inferred to simplify the calculation. In effect, if X, for example, is an arbitrary quantity which may be rendered as small as we would like, and we have an equation of this form:

$$A + BX + CX^2 + \text{etc.} = 0;$$

A, B, C, etc. being independent of X, this equation could not hold unless we have A = 0, B = 0, C = 0, etc., that is, without each term taken separately being equal to zero, whatever be the number of these terms. Now, by the same reasoning, if we had in general an equation of this form, P + Q = 0, such that P is a function of the given quantities or determined by the conditions of the problem, and on the contrary, Q is quantity we can suppose as small as we want, we would have necessarily P=0 and Q=0; but such is precisely the nature of the equation found above:

$$\left(\frac{TP}{y} - \frac{y}{a-x}\right) + \left(\frac{T'T}{y} - \frac{yMZ + aRZ - xRZ}{(a-x)(2a-2x-MZ)}\right) = 0.$$

Thus each of the terms of this equation, taken separately, is equal to zero; so we would be able to neglect during the course of the calculation the quantities T'T, MZ, and RZ – which do not enter in the first of these terms – without altering this first term; thus infinitesimal analysis differs from the method of indeterminates, only when in the first we treat as null, or rather we infer during the course of the calculation quantities which destroy themselves in the result if we allowed them to remain; instead in the method of indeterminates, we await the end of the calculation in order to remove the arbitrary quantities which must be eliminated. This last method could be substituted easily enough for infinitesimal analysis without employing the aid of imperfect equations, and without ever committing any error in the course of the calculation.

38. There is yet another means to substitute infinitesimal analysis by ordinary algebraic calculation; it is the method of limits or last ratios. For, although this method is founded entirely on the properties of limits and last ratios, it differs however from what

we name properly the method of limits, in that neither the quantities we have named infinitesimals, nor even their ratios, would enter separately in the calculation, but only the last values of these ratios, which being finite magnitudes, make of this method, less a particular calculation, as I have come to say it, than a simple application of ordinary algebraic calculation.

Thus we are concerned, being limited in introducing to ordinary algebra, not infinitesimal quantities, but the last ratios of these quantities, to substitute for means which infinitesimal analysis furnishes in order to discover the properties, ratios and any relations of these magnitudes which compose a proposed system, and see that it is properly what we have called the method of limits.

In order to explain the foregoing and give to it some life, let us take yet again the example treated above.

It is clear, by what has been said (§9), that although MZ/RZ is not at all equal to TP/MP; however the first of these quantities differs ever less from the second, as RS is closer to MP, that is, that MZ/RZ = TP/MP is an imperfect equation; but (designating by L. the expression of the limit or of last value) L.MZ/RZ = TP/MP is a perfect equation, or rigorously exact.

Likewise we shall prove that L. MZ/RZ = y/(a – x) is also a perfect equation, or rigorously exact; equating therefore these two values of L. MZ/RZ, it follows that TP/MP= y/(a – x), or TP= $y^2/(a - x)$, as above. Also, neither the infinitely small quantities MZ and RZ separately, nor even their ratio MZ/RZ, exist any longer in the equation, but only the limit or last value L. MZ/RZ, which is a finite quantity.

39. If this method were always as easy to put into usage as ordinary infinitesimal analysis, it could appear preferable; for it would have the advantage of moving to the same results by a direct and always luminous route, instead of that which drives to the truth only after travelling, if it be permitted to speak in such a way, through the land of errors.

But it is necessary to acknowledge that the method of limits is subject to a considerable difficulty which does not appear in ordinary infinitesimal analysis; it is that the infinitely small quantities not being separable, as in this case, from one another, and these quantities being found together two by two, we are not able to enter the properties which pertain to any one of them in particular into combinations, nor subject them to equations where they encounter all the transformations which could aid in their elimination; and this difficulty is much less sensed in the very operations of the calculation, than in the propositions and arguments which prepare or provide for these operations.

40. It seems, by what we have said (§2) on the possible origin of infinitesimal analysis, that the quantities which we have named infinitely small, have received this classification, because we believed in the beginning that, for the success of the calculations which use them, there must be attributed to these arbitraries, values which were really less than anything which could fall upon the senses, and anything which the imagination could conceive; but a more reflected-upon metaphysics, has shown that this is useless, because the success of the calculation comes, not by the attenuation of the

arbitrary quantities, but uniquely by the compensation of errors that they occasion in the calculation.

In effect, we have seen in the above example that the procedures and results of the calculation were absolutely the same, whatever value we attributed to the infinitely small quantities MZ and RZ, and that by consequence the character of the quantities of this type do not consist in the reality of their smallness, but rather in their absolute indetermination, that is, in the property by which they have to remain arbitrary during the whole calculation, and so independent of the proposed quantities, that one is always able to take them to be as small as we want without changing the conditions of the problem.

The infinitesimal quantities, as I have already stated (§24), are not therefore chimerical beings, but simple variable quantities characterized by the nature of their limit, which is 0, for infinitesimal quantities, and 1/0, for infinitely large quantities. Thus we can successively attribute diverse arbitrary values to these indeterminates, and likewise to all other indefinite quantities, and among these values, we must add the last of all which is 0 for infinitely small quantities, and 1/0 for infinite quantities.

41. This observation gives rise to the division of mathematical infinity into two kinds; to wit, the *sensible* or *assignable* infinite, and the *absolute* or *metaphysical* infinite, which is nothing other than the limit of the first.

So if we assign to any infinitely small quantity a determined value which is not 0, this value would be what I call a *sensible* or *assignable* infinitely small quantity, which I shall designate also by the name of *infinitely small;* instead if this value is the least of all, that is, if it is absolutely 0, it would be what I call an *absolute* or *metaphysical* infinitely small quantity, and what I would designate also by the name of a *vanishing* quantity.

Thus, a vanishing quantity is not what we call in general an infinitely small quantity, but only the last value of this quantity; it is, I say, only a determined value that we can attribute, as any other, to this arbitrary magnitude which in general we name infinitely small.

42. The consideration of these vanishing quantities would be almost useless, if we were confined to treating them as simply null quantities; for they offer only the vague relation of 0 to 0, which is not more equal to 2 as to 3 or to any other quantity; but it is necessary not to lose view of the point that these null quantities here have particular properties as last values of the indefinitely small quantities of which they are limits, and that we only give them the particular classification of vanishing in order to avoid the use of all the ratios and relations of which they are susceptible in the quality of null quantities, we would like to consider and enter into the combinations of our calculations only those which are assigned to them by the law of continuity, as we imagine the system of auxiliary quantities to be approaching by insensible degrees to the system of designated quantities: it is this that the great geometers understood themselves to be expressing in saying that the vanishing are considered quantities, not before they vanish, not after they have vanished, but at the very instant that they are vanishing.

In the above-treated case, for example, as long as RS does not coincide with MP, the fraction MZ/RZ is larger than TP/y; these two fractions become equal only at the moment where MZ and RZ are reduced to zero; it is true then that MZ/RZ is just as equal to any quantity besides TP/y, since 0/0 is an absolutely arbitrary quantity; but among the diverse values that one can attribute to MZ/RZ, TP/y is the only one that is subjected to and determined by the law of continuity; for if one constructed a curve with the abscissa being the indefinitely small quantity MZ, and the ordinate proportional to MZ/RZ, that which matches corresponds to the null abscissa, will be represented by TP/y, and not by an arbitrary quantity: for it is that which distinguishes the quantities that I call vanishing from those which are simply null.

Thus, although in general we have $0 = 2\times0 = 3\times0 = 4\times0 =$ etc., one cannot say of a vanishing quantity such as MZ, that MZ = 2MZ = 3MZ = 4MZ = etc.; for the law of continuity cannot assign between MZ and MZ a relationship other than equality, or any ratio other than that of identity.

43. We have seen that in introducing indefinitely small quantities into calculation, and by neglecting them in comparison with finite quantities, the equations would become imperfect, and the errors which had taken place were only compensated for in the sought result. We can now avoid, be we so inclined, this type of drawback by the means of vanishing quantities, which, being nothing other than last values of corresponding indefinitely small quantities, could, as all other values, be attributed to these indefinitely small quantities; and which, on another side, being absolutely null, can be neglected, when they are found added to such effective quantities, without the calculation ceasing to be perfectly rigorous.

44. Thus we can envisage infinitesimal analysis under two different points of view; in considering the infinitely small quantities either as effective quantities, or as absolutely null quantities. In the first case, infinitesimal analysis is nothing other than a calculation of compensating errors; and in the second, it is the art of comparing vanishing quantities among themselves and with others, to draw from these comparisons whatever relations and ratios exist among the proposed quantities.

As equal to zero, these vanishing quantities must be ignored in the calculation, as they find themselves added to or removed from some effective quantity; but, as we have just seen, they are nonetheless interesting relations to know, relationships which are determined by the law of continuity which the system of auxiliary quantities is subjected to in its changes. Now, to easily grasp this law of continuity, it is easy to perceive that one is obliged to consider the quantities in question at some distance from the term where they vanish entirely, if not they would only offer the indefinite relationship of zero to zero; but this distance is arbitrary and has as its only object to judge more easily the relationships which exist among these vanishing quantities: these are the relationships that one has in view in looking at infinitely small quantities as absolutely null, and not those which exist among quantities which are not yet brought to the term of their disappearance. These, which I have named indefinitely small, are not at all destined to themselves take part in the envisaged calculation from the point of view of which we are dealing at this moment, but employed only to aid the imagination, and to indicate the law of continuity which determines the ratios and any relationships among the vanishing quantities to which they correspond.

Thus, following this hypothesis, in the proportion MZ : RZ :: TP : MP, the quantities represented by MZ and RZ are indeed supposed to be absolutely equal to zero; but since it is their ratio that we need, it is necessary, to determine the equality of this ratio with TP / MP, to consider the indefinitely small quantities which correspond to these null quantities, not with the aim of introducing them into the calculation, but to bring them in under the denomination of MZ and RZ, the vanishing quantities which are their last values.

45. These expressions MZ, RZ thus represent null quantities, and we can employ them under the forms MZ, RZ, rather than under the common form 0, only because if we employed them in effect under this latter form, we could no longer separate, in the operations where they are found mixed, their diverse origins, that is, the diverse indefinitely small quantities to which they correspond. Therefore, the consideration of this, at least mentally, is necessary to grasp the law of continuity which determines the sought-for ratio of vanishing quantities that they have for limits, and by consequence it is essential not to lose sight of them and instead to characterize them by expressions which prevent confusing them.

46. The vanishing quantities which are the subject of infinitesimal calculus envisaged under this new point of view, are in truth entities of thought; but that does not prevent their having mathematical properties, nor their comparison as well as the comparison of the imaginary quantities that no longer exist; for it is also true to say, for example, that 60 = 20 + 40 as $\sqrt{-a} = \sqrt{-b} \times \sqrt{\frac{a}{b}}$. So no one revokes the exactitude of the results by the calculation of imaginaries, although they be algebraic forms, and hieroglyphs of absurd quantities; we cannot give exclusion to vanishing quantities which are at least limits of effective quantities, and touch so to speak, existence. What matters in effect is whether these quantities are or are not chimerical beings, if their relations are not (chimerical); and that these relations are the only thing which interests us? We are entirely in control, in submitting to calculation the quantities which we have named infinitesimal, to regard these quantities as effective quantities, or as absolutely null; and the difference which is found between these two methods of envisaging the question, consists in that by regarding these quantities as null, the propositions, equations and whatever results, are always exact and rigorous, but by relating quantities which are entities of thought, and expressing relations which exist between quantities which do not exist themselves: instead of regarding the infinitely small quantities as something effective, the propositions, equations and whatever results properly have for subject veritable quantities; but these propositions, equations and results are false, or rather they are imperfect, and become exact at the end only by compensation of their errors, a compensation, which, however, follows necessarily and infallibly from the operations of the calculation.

47. The metaphysics, which was just exposed, easily furnishes responses to all the objections which were made against infinitesimal analysis, which several geometers have believed based on a faulty principle capable of leading them astray; but they have been overwhelmed, if I may express it so, by the multitude of marvels, and by the flashing bursts of truths which are brought out *en masse* by this principle.

These objections can be reduced to this: either the quantities that we have named infinitely small are absolutely null, or not; for it is ridiculous to suppose that there exist entities which take the middle ground between quantity and zero. Now, if they are absolutely null, their comparison leads to nothing, since the ratio of 0 to 0 is not more *a* than it is *b*, or any other quantity whatever; and if they are effective quantities, we cannot without error treat them as null, as the rules of infinitesimal analysis prescribe.

The answer is simple: quite far from having the power, in effect, to consider infinitely small quantities, either as something real, nor as nothing, we could say on the contrary that we can at will regard them as null or as veritable quantities; for those who would want to regard them as null, could respond that what they name infinitely small quantities are not at all null quantities, but null quantities assigned by a law of continuity which determines the relation between them; that among all the relations of which these quantities are admitting of consideration as zero, they consider only those which are determined by this law of continuity; and that finally these relations are not at all vague or arbitrary, since this law of continuity does not at all assign, for example, several different relations to the differentials of the abscissa and of the ordinate of a curve when these differentials are vanishing, save one only, which is that of the sub-tangent to the ordinate. On the other side, those who regard infinitely small quantities as veritable quantities, could respond that what they call infinitely small is only an arbitrary magnitude independent of the proposed quantities; and that furthermore, we can, without supposing it null, nonetheless treat it as such without there ensuing any error in the result, since that error, if it existed, would be arbitrary like the quantity which would have occasioned it. Therefore, it is evident that a similar error could only exist between quantities of which at least one is arbitrary. Thus when we have reached a result which no longer contains any, and which expresses any relation between the given quantities and those which are determined by the conditions of the problem, one can assure that this result is exact, and that by consequence the errors which must have been committed in expressing these conditions, have been compensated for, and disappear, by the necessary and infallible succession of the operations of the calculation.

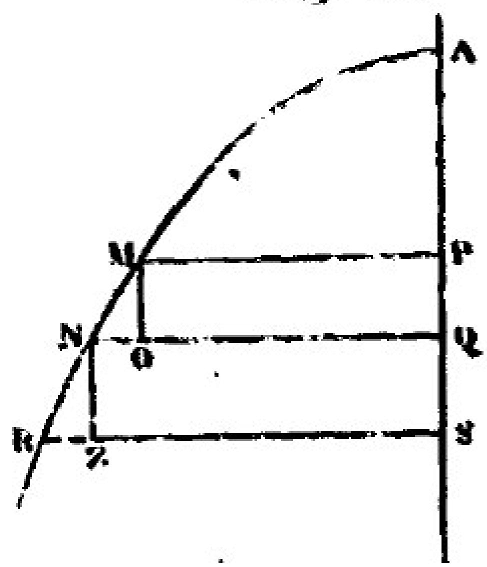
48. Other geometers, apparently blocked by the objection just discussed, have attached themselves to simply proving that the method of limits whose methods are rigorously exact in all points, necessarily leads to the same results as infinitesimal analysis. However, in admitting that the principle of this method is very luminous, we may be concealing that it only evades the difficulty without resolving it; that the method of limits leads to the results of infinitesimal analysis only by a difficult and roundabout route; and that finally this method, far from being the same as that of the infinitesimal calculus, is, on the contrary, only the art of doing without it, and supplanting it by ordinary algebraic calculation: which we would succeed in doing in a more simple manner, it seems to me, with the method of indeterminates. But why would one adopt one of these methods to the exclusion of the others, since they can aid each other mutually? Thus let us employ everything together, infinitesimal analysis properly stated, the method of limits, and that of indeterminates, following the indicated circumstances, and neglecting no other means which can lead us to a knowledge of the truth, or in simplifying the search.

It now remains for me to show by several examples the application of the general principles which I have just explained; and that is what I am going to do in giving an idea of differential and integral calculus, which are, properly speaking, infinitesimal analysis itself put into practice.

49. If one were to attribute successively to the same variable quantity two values differing infinitely little from each other, the difference between the second of these two values and the first shall be named the *differential* of this first value.

Let, for example, AMN (*Fig. 2*), be a curve relative to which we have some question to resolve, such that the ordinate MP is one of the quantities designated by this question. I suppose, moreover, that in order to facilitate the solution, we could draw parallel to MP, and at an arbitrary distance from this ordinate, an auxiliary line NQ, and next let this line approach MP continuously until it coincides with it; the line NO, or NQ – MP would be (§19) an infinitely small quantity. Yet, as it is the difference of two values attributed successively to the ordinate, NQ and MP, it is suitable to designate it in the discourse by the diminutive expression of differential of the variable MP, and to represent it in the calculation by this same variable preceded by the characteristic *d:* thus, in giving the name *y* to the ordinate MP, *dy* signifies the same thing as the differential of MP.

Fig. 2.



But suppose, as we have done, that NQ approached MP perpetually, that is, suppose that AQ also approaches AP perpetually; for the first of these two suppositions necessarily brings about the second; thus in giving the name x for the abscissa AP, PQ or MO will be the differential of x, and we would have MO=dx at the same time that NO=dy.

If, moreover, we were to suppose NQ= y' and AQ= x', we would have $y'= y + dy$ and $x'=x + dx$; that is to say, that the differentials dy and dx, are none other than the increase of the corresponding variables y and x, or the quantities by which they are augmented as they become y' and x'.

50. Now let a new value RS be attributed to the ordinate, such that PQ and QS differ infinitely little from one another, or have for a last ratio a ratio of equality; in order that this be, it is obviously necessary, since NQ by the first hypothesis is already supposed to approach MP perpetually, it is necessary, I say, that RS also approach the same line MP perpetually, in a manner that it finishes as NQ does by coinciding with it; otherwise it is clear that the ratio of QS to PQ, which must by hypothesis be approaching unity without cease, without withdrawing from it: also the ratios of NQ to MP, of RS to MP, of RS to NQ and of QS to PQ, will all have for a limit the ratio of equality. Moreover, it is visible that due to the law of continuity, the ratio of RZ to NO will be the same case. Thus, following the general notion we have given above of differential quantities, QS must be the differential of AQ, RZ that of NQ, QS–PQ or NZ–MO that of PQ, and finally RZ–NO that of NO; likewise NO or NQ–MP is that of MP. So, conforming to the convention made on the subject of the manner of expressing the differentials in the calculation, we must have QS = dx', RZ = dy', QS–PQ = d(MO), RZ–NO = d(NO). However we have

already found MO = dx, NO = dy; thus QS–NO = ddx, RZ–NO = ddy; that is, the quantities ddx and ddy (we can also write it in the manner d^2x, d^2y), will be the differentials of differentials of x and y, and are what, to be brief, we name *second differentials*, or *second order differentials*; that is, ddx is the differential of the second order, or the second differential of x, and ddy that of y.

Now, since QS and PQ are supposed to differ infinitely little from one another, their difference ddx is infinitely small relative to each of them (§28). So, differences of the second order are infinitely small relative to the first differentials or those of the first order.(*)

51. We could differentiate similarly in their turn differentials of the second order, and this differentiation would result in differentials of the third order; differentiating this would result in that of the fourth order, and so forth: in a manner $dddy$, or d^3y, will be the third differential of y; $ddddy$, or d^4y, the differential of the fourth order, etc. Therefore, according to what we have just said on the generation of differentials of the first and second order, it is easy to comprehend how the higher orders are made; so I will not stop there; I will only say that in attributing for each new order a new auxiliary value to each of the variables, such that, not only each of these new values differ infinitely little from that which precedes it, but the same thing takes place between their differentials, the differentials of their differentials, and so forth.

52. To *differentiate* a quantity is to assign its differential; that is, if X, for example, is any function of x, to differentiate it will be to assign the quantity by which this function will increase while supposing that x augments by dx.

To *integrate* or to sum a differential, on the contrary, is to return to this differential the quantity which had produced it by differentiation, and this latter quantity is called the *integral* or *sum* of the proposed differential: so x, for example, is the integral or the sum of dx, and to integrate or sum dx is nothing other than to assign this quantity x which is its sum or integral.

We have seen that the differential of a quantity is expressed in the calculus by this same quantity preceded by the characteristic *d;* reciprocally, it is conventional suitable to express the integral or sum of any differential by this same differential preceded by the characteristic $\int$*:* that is, that $\int$dx, for example, signifies the same thing as the sum of *dx:* thus we clearly have $x = \int dx$.

53. We call the *differential and integral calculus* the art of finding any ratios and relationships which exist between the proposed quantities, by aid of their differentials. The name *differential calculus* is properly applied to the art of finding the ratios or relationships of the differential quantities, and then to eliminate them by the ordinary rules

(*) §50 – If instead of drawing the new auxiliary line RS in a manner that the lines QS and PQ differ infinitely little from one another, we could draw it such that QS be precisely equal to PQ, that is, such that AP, AQ, AS, are in arithmetic progression, we will have $ddx = 0$, or dx constant: thus we could suppose one of the differentials constant; but from AP, AQ, AS being in arithmetic progression, it does not follow that MP, NQ, RS, are so also, unless the line AMN be right: thus, from supposing ddx equal to zero, it would not follow that we would also have $ddy = 0$.

of algebra, and that of *integral calculus* the art of integrating or eliminating these same differential quantities by processes which indicate a return of a differential to its integral.

My aim here is not to write a treatise on the calculus; but only to indicate the fundamental rules thereof, and to show that these rules are nothing other than an application of the general principles just exposed.

54. Therefore, we propose, at first, to assign the differential of the sum x+y+z+ etc, of several variables.

By hypothesis x becomes $x+dx$, y becomes $y+dy$, etc. Thus the proposed sum becomes $x+dx+y+dy+z+dz$+etc., and it increases by $dx+dy+dz$+etc., and this increase is precisely what we have named differential.

55. We now require the differential of $a+b+c+$ etc. $+x+y+z$+etc. : a,b,c, etc. being constants, and x, y, z, etc. being variables. By hypothesis, a remains a, b remains b, c remains c, etc., x becomes $x+dx$, y becomes $y+dy$, etc. Thus the proposed sum becomes $a+b+c,$ etc. $+x+dx$+etc.; thus it is increased by $dx+dy+dz+$ etc., and this increase is the sought differential; thus this differential is the same as if there had not been any constants in the proposed sum.

We require the differential of ax.

By hypothesis, a remains a, and x becomes $x+dx$. Thus ax becomes $ax+adx$; so it is increased by adx, and this increase is the sought differential.

56. We require the differential of xy.

We see by the preceding that it is $ydx+xdy+dxdy$, that is to say, we have $d.xy= ydx+xdy+dxdy$.

However, I observe, with regard to this equation, that dx and dy being infinitely small relative to x and y, the last term $dxdy$ is itself infinitely small relative to each of the others; in other words, that the quotient of this last term over each of the others, is an infinitely small quantity. So if we were to neglect it in the preceding equation, it would become then $d.xy= xdy+ydx$, this equation will be what I have named an imperfect equation. However, since imperfect equations can (§31, §34) be employed as rigorous equations, without there ensuing any error in the sought result, it is evident that I may make use of this latter equation in place of the former; and as it is simpler, I will on this occasion, abridge and facilitate, with its aid, the operations of my calculation.

Thus, I will say that the differential of a quantity, which is the product of two variables, is equal to the product of the first variable and the differential of the second, plus the product of the second variable by the differential of the first; and this proposition would be among those which I have called (§35) imperfect propositions, that is, susceptible of being conveyed by an imperfect equation, and as such, leading towards exact, rigorous results.[*]

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[*] §56 – If from the imperfect equation $dxy=xdy + ydx$, I wanted to create a rigorous equation, I could do so, by first restoring to the right-hand-side the missing term $dxdy$; but I could also do so in the following manner: I would divide everything by dy, for example, and I would have the new imperfect equation

57. We will find by procedures like those above, that we have the imperfect equation *d.xyz* = *xy dz* + *xz dy* + *yz dx*.

We would similarly find the imperfect equation *d. x/y* = (*ydx* - *xdy*)/ *yy*.

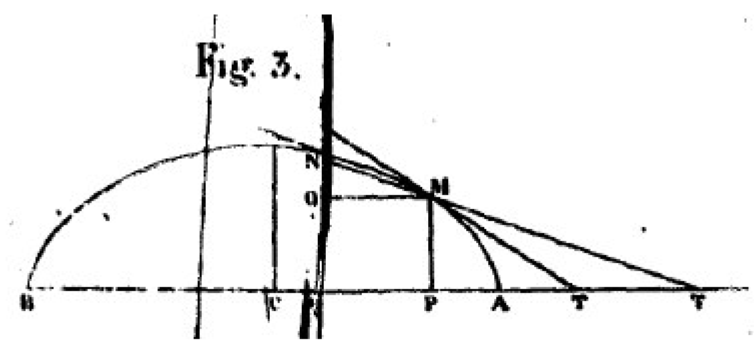
We would similarly find the imperfect equation $d.x^m = mx^{(m-1)}\, dx$, etc.

58. These are the principal rules of differential calculus; let us now proceed to the integral calculus, which is the inverse method.

1. Since the differential of *x* is *dx*, the integral of *dx* would be *x*, that is, one would have $\int dx = x$. However since the differential of *a*+*x* is also *xdx* (§55), it follows that the integral of *dx* is just as much *a* + *x* as it is *x* alone, and that in general each differential has as many diverse integrals as we would want to give it; but all these integrals differ only by a constant quantity. Thus it suffices to determine any one to add to it an arbitrary constant to represent all the others: that is, that all the possible integrals of *dx* will be represented by *x* + A, A being an arbitrary constant.
2. Since the differential of *x*+*y*+*z*+ etc. is *dx*+*dy*+*dz*+ etc., the integral of this differential will be *x*+*y*+*z*+ etc. + A, A being an arbitrary constant.
3. The differential of *xy* being *xdy*+*ydx* (§56) as well as that of *xy* + A, the integral of *xdy* + *ydx* is reciprocally *xy* +A, A being an arbitrary constant.
4. We shall likewise find that the integral of (*ydx*–*xdy*)/*yy* is *x*/*y* + A.
5. We shall likewise find that the integral of $mx^{m-1}\, dx$ is x^m + A, etc.

Such are the principal rules of integral calculus; it now remains for us to demonstrate by several particular examples the application of these rules and those of differential calculus; this will be done as succinctly as is possible for us.

FIRST PROBLEM



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d.xy / dy = *y*(*dx*/*dy*) + *x*; and since (§19) an auxiliary quantity differs infinitely little near its limit, I can, in the preceding equation, put *lim.* (*d.xy / dy*) in place of *d.xy / dy* and *lim.* (*dx*/*dy*) in place of *dx*/*dy*, without the equation ceasing to be imperfect (§32). So it then becomes *lim.* (*d.xy / dy*) = *y* × *lim.* (*dx*/*dy*) + *x*; but every limit is by the same definition (§17) a designated quantity. Thus since *dx* and *dy* are auxiliaries, *lim.* (*d.xy / dy*) and *lim.* (*dx*/*dy*) are the designated quantities; therefore all the terms of the preceding equation *lim.* (*d.xy / dy*) = *y* × *lim.* (*dx*/*dy*) + *x*, are designated quantities; and therefore (§34) this equation is necessarily and rigorously exact.

59. Given an elliptical curve AMB (Fig. 3), find the sub-tangent TP which corresponds to a given point, M, of this curve.

Let AB be the major axis of the curve: let us call *a* half of the major axis, *b* the semi-minor axis, *x* the abscissa AP, and *y* the ordinate PM; thus we will have $yy = \frac{bb}{aa}(2ax - xx)$. This posed, let us draw a new ordinate NQ infinitely close to MP, such that, this auxiliary line NQ is at first taken to an arbitrary distance from MP, and is then imagined to be continuously drawing nearer to it (MP), in a manner in which the last ratio, is an ratio of equality; the lines MO, NO would be (§49) the respective differentials of *x* and *y*. Now the similar triangles TPM, MZO give $\frac{TP}{MP} = \frac{MO}{ZO} = \frac{MO}{NO + ZN}$. However it is evident that the more NQ approaches MP, the more ZN diminishes relative to NO, and that their least ratio is 0. So ZN is infinitely small relative to NO; thus TP/MP=MO/NO is an imperfect equation (§31); that is, TP/*y*=dx/dy is an imperfect equation.

From the other side, the equation of the curve being $yy = \frac{bb}{aa}(2ax - xx)$, we would have in differentiating it, another imperfect equation $ydy = \frac{bb}{aa}(adx - dx)$; thus substituting in this latter the value of dx taken from the former, and reducing, we will have $TP = \frac{aa}{bb} \times \frac{yy}{a - x}$; an equation which no longer contains infinitesimal quantities, and is necessarily and rigorously exact (§34).

60. Another solution: let us consider the proposed curve as a polygon with an infinite number of sides; that is, in place of the proposed curve, take a polygon with a number of sides, and next suppose that this number of sides increases perpetually, in such a manner that the least ratio of this polygon with the curve is one of equality. As it is absolutely impossible that the curve can be considered exactly like a polygon, the equations by which I would express the conditions of the problem following this hypothesis, will not be exact; but since the polygon is supposed to approach the curve without cease, the errors which will be found in these equations, will be attenuated as much as we would like, therefore, these same equations would be that which I have called imperfect.

So the triangles T'MP, MNO give me the equation T'P/(MP) = MO/NO; substituting TP for T'P, which differ infinitely little, we would have this imperfect equation, TP/MP=MO/NO or TP/y=dx/dy, the same as that we found above, and which, combined with that of the curve, gives me the same result.

61. We could also, were we so inclined, apply the method of indeterminates to this question, without changing anything in the procedures of the calculation. In effect, after having found the two imperfect equations $\frac{TP}{y} = \frac{dx}{dy}$ and $2ydy = \frac{bb}{aa}(2adx - 2xdx)$, I add mentally to one of the former members, a quantity φ; I introduce similarly into the second a quantity φ' which renders it (similarly) rigorously exact: the agreed upon quantities φ

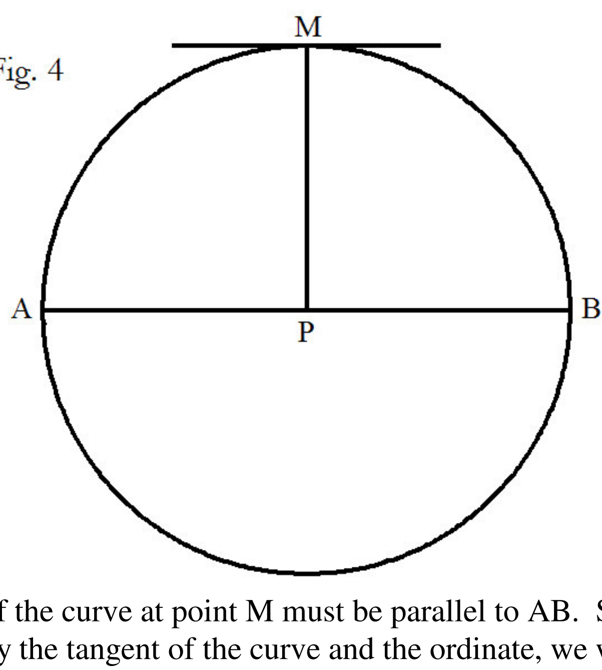
and φ' are infinitely small relative to those to which we add mentally. This posed, I compare the two preceding equations without taking into account φ and φ'; the equation $TP = \frac{aa}{bb}\frac{yy}{a-x}$ which shall result, being possibly not exact, I add again mentally a quantity φ'' which makes it so. However as this quantity φ'' can only be infinitely small, I soon recognize that it is absolutely null, because the other terms of the equation no longer contain infinitesimal quantities; so putting all the terms together, the equation which would be $TP - \frac{bb}{aa}\frac{yy}{a-x} + \varphi'' = 0,$ could not hold following the method of indeterminates, without any of its terms in particular being equal to zero: thus $\varphi''= 0$, and $TP = \frac{bb}{aa}\frac{yy}{a-x}$, as above.

62. In general, it is clear according to what has just been said, that if we name P the sub-tangent of any curve, we would have the imperfect equation P= y dx/dy; thus (§34) we would have the rigorous and exact equation P=y × lim(dx/dy).

If we call Q the angle swept out between the tangent of the curve at any point and its corresponding ordinate [angle TMP], evidently we will have, tan Q=P/y and cot Q= y/P; thus we will have the imperfect equations tan Q= dx/dy and cot Q= dy/dx, or the rigorous equations, tan Q= lim (dx/dy) and cot Q= lim (dy/dx).

SECOND PROBLEM

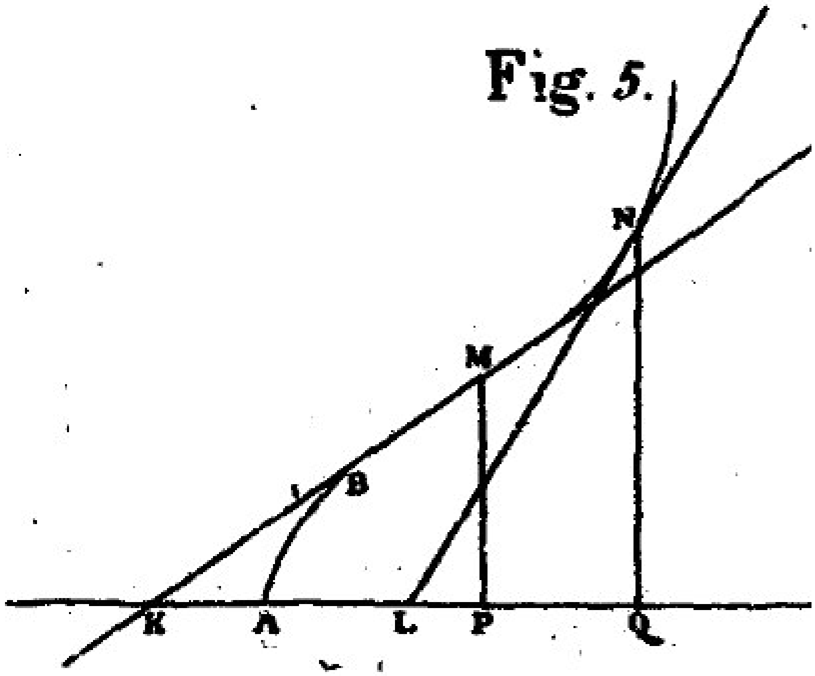
Fig. 4



63. We require the value which must be attributed to x so that the function $\sqrt{2ax - xx}$ be a *maximum*, that is, larger than we have attributed to x at another arbitrary value. Let $\sqrt{2ax - xx} = y$ or $yy = 2ax - xx,$ and let us construct a curve of which the abscissa is x and ordinate y, thus the question would be to find the largest ordinate of this curve. Let AMB (Fig. 4) be this curve and MP its largest ordinate: that posed, since including the point M the other ordinates decrease, either from side A, or from side B, it is clear that the sub-tangent of the curve at point M must be parallel to AB. So naming Q, as above, the angle formed by the tangent of the curve and the ordinate, we would have at point M, cot Q= 0, or (§62) lim (dy/dx) = 0. Thus, I differentiate the equation of the curve, and I have the imperfect equation ydy= adx - xdx, or dy/dx=(a – x)/y, thus I have the rigorous equation lim (dy/dx)= (a-x)/y, or cot Q=(a-x)/y. So we must have cot Q=0; thus (a-x)/y =0 or finally a=x. Q.E.D.

64. The procedure to follow in order to find the largest ordinate of any curve, is thus to differentiate the equation, extracting the value of lim (dy/dx), and equating it to zero. We could enunciate this rule commonly by saying simply that one must differentiate y and equate dy to zero; but if this enunciation is shorter, it is also less exact.

THIRD PROBLEM



65. Given a proposed curve having an inflection point, determine the abscissa or ordinate to which it returns. Let ABMN (Fig. 5) be the proposed curve; let AB be the abscissa, and MP the corresponding ordinate at the sought inflection point M; now draw a tangent line MK at this point of inflection. It is clear that the angle KMP[†] is a *minimum*, that is, less than the angle LNQ formed by another tangent NL, and the corresponding ordinate NQ; thus the tangent of angle KMP[†] is Also a *minimum*, and its cotangent a *maximum*; but this cotangent is in general (§62) lim (dy/dx): se we must find (§63) $\lim\left(\frac{d.\lim(\frac{dy}{dx})}{dx}\right) = 0$. Q.E.D.

Let, for example, $b^2y = ax^2 - x^3$ be the equation of the proposed curve. I differentiate, and I have the imperfect equation $b^2dy=2axdx-3x^2dx$, or the rigorous equation $\lim\left(\frac{dy}{dx}\right) = \frac{2ax-3x^2}{b^2}$, thus $\frac{2ax-3x^2}{b^2}$ must be a *maximum*, or $\lim\left(\frac{d(2ax-3x^2)}{dx}\right) = 0$; that is, we must have 2a – 6x= 0, or x= ⅓a.

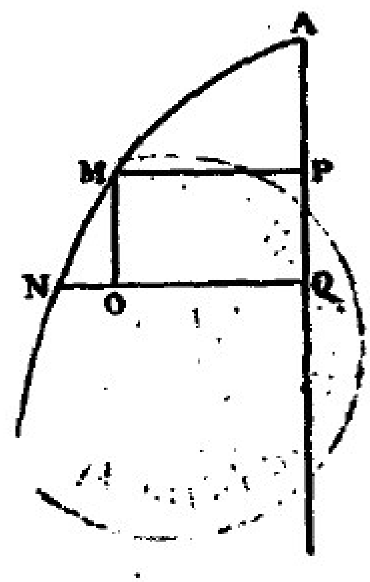
[†] This is Carnot's terminology, which is different from our own. Today, this would be angle MKP, not KMP.

FOURTH PROBLEM

66. Find the surface of a parabolic segment.

Let AMP be this segment (Fig. 6); if we suppose that the abscissa AP increases by an infinitely small quantity PQ, this segment will increase at the same time as the quantity MNPQ; that is, PQ being supposed as the differential of x, MNPQ will be the differential of the sought segment. Reciprocally, the sought segment is the integral of MNPQ, that is, we have AMP=∫(MNPQ); but if we lower MO perpendicularly to NQ, it is evident that the last ratio of space MNO to space MOPQ is zero; thus the first of these spaces is infinitely small with regard to the second; so we have the imperfect equation MNPQ= MOPQ. Substituting therefore the second of these quantities into the first, in the exact equation AMP = ∫(MNPQ), we will have the imperfect equation AMP=∫(MOPQ), or AMP =∫ydx; but the equation of the curve is, calling P its parameter, yy=Px, from which we have the imperfect equation dx=(2ydy)/P; in then substituting for dx, in the first of these two imperfect equations, its value derived by the second, we would have a new imperfect equation AMP= $\int((2y^2dy)/P)$. However (§58), we have $\int((2y^2dy)/P)=(2/3)y^3/P$, so AMP = $(2/3)y^3/P$, an equation which, containing now only designated quantities, and can only be rigorously exact: which was to be found.

Fig. 6.



The same method applies itself evidently to the quadrature of any other curve, and for analogous reasons, it is comforting to be able to extend this to their rectification and to the investigation of arbitrary solids.

68. This small number of examples should suffice in order to make known the character of infinitesimal analysis. In vain, say the adversaries, it is the certain ruin of mathematics which admits errors, as was done, by employing imperfect equations. These errors could have dangerous consequences, since we have infallible means to make them disappear, and certain signs to know when they have disappeared? Shall one renounce the immense advantages that this calculus procures, for fear of deviating one instant from rigorous procedures of elementary geometry, a thorny path where it is so difficult not to be lead astray, or, shall one prefer a singular and easy route by which this analysis leads to discoveries? Such is that which the method of limits offers, when one wishes to employ it exclusively. For those who wish to banish the notion of infinitesimal quantities, are reduced, either to supplant it by common algebra, the which present countless difficulties, or to continually make use of the names, infinite or infinitely small, while at the same time denigrating them, if it may be expressed thusly, and which concerns the chimerical existence of similar things, of which they are hieroglyphics. We employ, one may say,

these terms only figuratively; but I ask if a language, figurative and abstract, is that which is convenient to the simplicity of mathematics, and above all to this rigor which one wants to support to condemn the theory of the infinite. Do these two methods not come to the same thing, or rather, are they not the same methods employed in diverse ways? In a word, are these not always the same ideas to formulate, the same relations to express? Thus, why not formulate these ideas, and express these relations in the most clear and simple manner?

E N D.

Corrections:
p.162 of original had "(2a – 2x) – MZ" where it should have had "(2a – 2x – MZ)"

Section 37 – infer is really "sous-entendre"

Section 42 – See Riemann for Anti-Dummies 59

Section 44 – th end could be looked at again

There only is one chapter in the 1797 paper. The preface is not actually from the 1797 paper. I should add the abstract.