Solution of determined cubic Equations through the so-called Cardan's Formula

699. **Problem.** To find an irrational expression for the roots of the the cubic equation $x^3 = ax + b$. Every cubic equation can be reduced to this form (282).

Solution. Set $x = \sqrt[3]{A} + \sqrt[3]{B}$, and further suppose, A and B should both be roots of the following quadratic equation, $z^2 = \alpha z - \beta$. In order to determine α and β by means of the given coefficients a; b; I consider $A + B = \alpha$, $AB = \beta$ (224), and

$$\begin{aligned} x^3 &= A + 3\sqrt[3]{A^2B} + 3\sqrt[3]{AB^2} + B \\ &= A + B + 3\sqrt[3]{AB}(\sqrt[3]{A} + \sqrt[3]{B}) \\ &= 3x\sqrt[3]{AB} + (A+B). \end{aligned}$$

From this follows, $a = 3\sqrt[3]{AB} = 3\sqrt[3]{\beta}$, $b = A + B = \alpha$. Consequently, $z^2 = bz - \frac{a^3}{27}$. Hence, $z = \frac{1}{2}b \pm \sqrt{\frac{1}{4}b^2 - \frac{a^3}{27}}$. May the irrational part be designated c, then $A = \frac{1}{2}b + c$, $B = \frac{1}{2}b - c$, and $x = \sqrt[3]{\frac{1}{2}b + c} + \sqrt[3]{\frac{1}{2}b - c}$, instead of which I shall write $m + n$.

700. **Corollary.** Dividing the cubic equation with $x - m - n$, (226) yields the final remainder $-b - a(m+n) + (m+n)^3$. However, it was previously found that $[x^3 =]$ $(m+n)^3 = A + B + 3\sqrt[3]{AB}(\sqrt[3]{A} + \sqrt[3]{B}) = b + a(m+n)$. Thus, adding $-b - a(m+n)$ cancels out. The quotient then is $x^2 + (m+n)x + (m+n)^2 - a$. Here $3mn = a$, therefore, $(m+n)^2 - a = m^2 + n^2 - 3mn$. Now the roots are $\frac{m + n \pm \sqrt{(m+n)^2 - 4(m^2 + n^2 + 3mn)}}{2}$. Therefore, these roots become $x = \frac{m + n \pm (m-n)\sqrt{-3}}{2}$.

701. **Corollary.** If c is possible, then m and n are possible, therefore two roots of the cubic equation (699) are impossible.

702. **Theorem.** Let K be a possible magnitude, λ an impossible one, whose square is possible. For example, $\lambda = L\sqrt{-1}$ where L is possible. Further, P, Q should also designate possible magnitudes. K, L, P, Q may now be rational or irrational magnitudes. Then, **I**) $\sqrt[3]{K + \lambda} = P + Q\sqrt{-1}$; **II**) $\sqrt[3]{K - \lambda} = P - Q\sqrt{-1}$.

Proof. It is immediately apparent that the cubic root of the sum of a possible and impossible magnitude is impossible: however, it must also have a possible part, for an impossible magnitude alone cannot produce the possible part K in its cube. Taking the cube of both sides of **I**) and setting the possible equal to the possible and the Impossible equal to the impossible (approximately according to the manner concluded in 235; 233), then K = $P^3 - 3PQ^2$ and λ = $(3P^2Q - Q^3)\sqrt{-1}$ or L = $3P^2Q - Q^3$. Consequently, $K - \lambda$ = $P^3 - 3P^2Q\sqrt{-1} - 3PQ^2 + Q^3\sqrt{-1}$ = $(P - Q\sqrt{-1})^3$. Therefore it becomes evident that the equations **I**) and **II**) are real together, or that the second follows from the first. It is only still to be proven, that *P, Q* are possible magnitudes. This manifests itself in the following form: $(K+\lambda)(K-\lambda)$ = $K^2 - \lambda^2$ = $K^2 + L^2$ which is certainly a possible magnitude. Therefore its cubic root is as well. Denote this cubic root *e*. Then from the course of the proof, $(K+\lambda)(K-\lambda)$ = $((P+Q\sqrt{-1})(P-Q\sqrt{-1}))^3$ = $(P^2+Q^2)^3$. Therefore e = $P^2 + Q^2$ or $Q^2 = e - P^2$. Substituting this into the equation K = $P^3 - 3PQ^2$ gives **A**) $P^3 - \frac{3e}{4}P = \frac{1}{4}K$; then, similarly setting $P^2 = e - Q^2$ in L = $3P^2Q - Q^3$ gives $Q^3 - \frac{3e}{4}Q = -\frac{1}{4}L$, where *P* and *Q* each have a certain possible value (272). For the proof it is sufficient, that it gives a possible value for each of the magnitudes. If they still have impossible [values], it is negligible. The proposition is true in the very sense in which it was maintained, such that the cubic root of a possible magnitude is possible, although this cubic root always still has two impossible values (240).

703. Corollary. Even the only cited [angeführte] consideration shows, what both of the remaining values of **P** in **A** are. Namely, from 240 also, $\sqrt[3]{K+\lambda}$ = $\frac{(P+Q\sqrt{-1})(-1\pm\sqrt{-3})}{2}$ = $-\frac{1}{2}P \mp \frac{1}{2}Q\sqrt{3} \pm \frac{1}{2}P\sqrt{-3} - \frac{1}{2}Q\sqrt{-1}$. So, the root in **A** signifies the possible part of the cubic root of $K+\lambda$, which is also, therefore, signified as $-\frac{1}{2}P \mp \frac{1}{2}Q\sqrt{3}$.

Example. If $K = 81$; $\lambda = 30\sqrt{-3}$, then $n = 21$ and $P^3 - \frac{63}{4}P = \frac{81}{4}$; Here $P = -3$, $Q = 2\sqrt{3}$ or $(-3+2\sqrt{3})^3$ = $81 + 30\sqrt{-3}$, just as can easily be affirmed through the Trial, if it is also not known, as *P* and *Q* are found. Thus, the roots of the equation **A** are -3, $-\frac{3}{2}$, and $\frac{9}{2}$. It becomes apparent, on this account, that *Q* is determined by a cubic equation.

704. Remark. Were the shortening of the calculation by the use of the magnitude *e* not followed, as *Clairaut* taught in the German translation of his *Algebra* 256, then, removing *Q* (199) from both of the equations, which, in the beginning of the proof, were set *K* = ..., and *L* = ..., thereupon an equation of the 9th degree for *P* results, which can be reduced to a cubic equation. Landen arrives at this in *Mathematical Lucubrations* pg. 54.

705. Theorem. If *c*, 699, is impossible, then *m* and *n* are also impossible (702), and thus the three roots of the cubic equation 699 are possible.

Proof. Thereupon $m = P + Q\sqrt{-1}$, $n = P - Q\sqrt{-1}$ (702). Consequently, $m + n = 2P$; $m - n = 2Q\sqrt{-1}$, which, according to 700, is multiplied by $\sqrt{-3}$, giving - $2Q$ x 3. Therefore the three roots are $2P$ (699) and $P \mp 3Q$ (700).

706. Corollary. Since c is possible or impossible according to whether $\frac{1}{4}b^2 - \frac{1}{27}a^3$ is positive or negative, then the cubic equation (699) has purely possible roots, or two impossible roots, according to whether $\frac{1}{4}b^2 - \frac{1}{27}a^3$ is negative or positive [respectively].

707. Corollary. If the cubic equation has two impossible roots, then, from the procedure of 699, $m + n$ yields a possible value for the remaining possible root.

708. Corollary. Were, however, all three of the roots of the cubic equation possible, **m** and **n** would become impossible, such that, therefore, the just mentioned procedure would not, as it seems, be able to establish three possible roots. This apparent difficulty shall be lifted at once.

709. Theorem. If the cubic equation has three possible roots, then the procedure of 699 produces all three.

Proof. Thereupon, $\sqrt[3]{A} = P + Q\sqrt{-1}$ and also $= (P + Q\sqrt{-1})\frac{-1 \pm \sqrt{-3}}{2}$. Similarly [imgleichen], $\sqrt[3]{B} = P - Q\sqrt{-1}$ and also $= (P - Q\sqrt{-1})\frac{-1 \pm \sqrt{-3}}{2}$ (240). Now, should $3\sqrt[3]{AB} = a$ be possible, then, in order to constitute x, whose product is possible, such expressions of the cubic roots must be taken together. Consequently, the three values of **x** are:

1	$P + Q\sqrt{-1} + P - Q\sqrt{-1} = 2P$
2 3	$(P + Q\sqrt{-1})\frac{(-1 \pm \sqrt{-3})}{2} + (P - Q\sqrt{-1})\frac{(-1 \mp \sqrt{-3})}{2} = -P \mp Q\sqrt{3}$

If the upper sign of the impossible root is taken for the second value of x, the lower being used for the third, then $\sqrt[3]{AB} = (PP + QQ)\left(\frac{-1-(-3)}{2}\right)$ obtains a possible value. Conversely, this product and its cubic root would become impossible, were $(P + Q\sqrt{-1})\frac{(-1 \pm \sqrt{-3})}{2} + (P - Q\sqrt{-1})\frac{(-1 \pm \sqrt{-3})}{2}$ assumed.

710. Theorem. If the cubic equation has two impossible roots, then the procedure 699 produces such, in addition to the possible root (707).

Proof. Thereupon, according to 240, x also obtains both of the following values in addition to the value $m + n$ consisting of two possible parts:

$$\begin{array}{c|l} 2 \\ 3 & m\frac{(-1\pm\sqrt{-3})}{2}+n\frac{(-1\mp\sqrt{-3})}{2}=-P\mp Q\sqrt{3} \end{array}$$

from which becomes $\frac{-(m+n)\pm(m-n)\sqrt{-3}}{2}$. There, since **m** and **n** are not the same, the impossible part does not cancel itself out.

711. Corollary. If factors [Wurzelgleichungen] (220) are constructed from the three values of x (709), and multiplied, then, $x^3-3(P^2+Q^2)x-2(P^3-3PQ^2) = 0$ results. However, $P^2+Q^2 = \sqrt[3]{AB}$ (702) $= \frac{1}{3}a$ (699) and $P^3-3PQ^2 = K$ (702) $= \frac{1}{2}b$ (699), thus the equation transforms into $x^3-ax-b=0$, which is the equation of 699.

712. Corollary. The equation just found can also be expressed as $\mathbf{x}^3-3\mathbf{ex}-2\mathbf{K} = 0$. The roots of the equation A (702) are the halves of the roots of this equation (280).

713. Corollary. The equation 699 can also be formed out of the factors [Wurzelgleichungen] in 710. This account is found in *Maria Gaetana Agnesi, Instituzioni analitiche ad uso della gioventu' Italiana T. I.* § 187.

714. Corollary. If, in 710, $m = n$, then $c = 0$ in 699, and thus, both of the remaining values of x are each = - m and the equation, therefore, has two equal roots each = - m, the third being = $2m$.

715. Scholium. In this case $\frac{1}{4}b^2 = \frac{1}{27}a^3$ (699), consequently, $\frac{1}{2}b = \left(\frac{a}{3}\right)^{\frac{3}{2}}$. Moreover follows, the equation $x^3 = ax+\frac{2a}{3}\sqrt{\frac{1}{3}a}$ and its root, $2m = 2\sqrt[3]{\frac{1}{2}b} = \frac{2a}{3}\sqrt{\frac{1}{3}a}$. A completely simple example of such a case can be made in rational numbers if the triple of a square is taken for a.

716. Corollary. If the root of a cubic equation becomes expressed as $\sqrt[3]{\frac{1}{2}b+c} + \sqrt[3]{\frac{1}{2}b-c}$, then c is either possible or impossible according to whether two of the roots are impossible or all three roots are possible (701; 705), and hence, the true cause of why it becomes impossible in the latter case reveals itself. Namely, the adopted expression of the root, signifies all three roots, by virtue of the procedures 709; 710. However, it cannot signify three possible roots in the case that **c** is not impossible, (709) for if c were possible, it would yield two impossible roots according to 710.

717. Scholium. The procedure described in 699 is called **Cardan's Formula**. Cardan reports it in his Algebra, which carries the title: *Artis magnae siuede regulis algebraicis liber unus,* which waas published in Basel, 1570, together with his *opera nouo de proportionibus numerorum; motuum* etc. In Chapter I, Page 5 of this publication, he ascribes the discovery of the formula to Scipio Ferreus of Bononien. According to his expression it is called: *capitulum cubi, & rerum, numero aequalium*; which is, according to our manner of saying, $x^3 + gx = h$, viz. the old Algebraicists calling *rem* what we denote *x*. Nicolaus Tartalea[1] had also arrived at this formula, and imparted it to Cardan, though without proof, which Cardan had found and provided. Cardan's Formula expressed in letters, appears somewhat different than the form given (699), however, it can easily be brought into that form. Cartesius[2] had arrived at the formula which flows out of Cardan's in his Geometry (Book III, Page 92 of the Amsterdam Edition 183; or the Franckfurt 1695) and Franciscus van Schooten repeated it in the work *de cubicarum aquat. resolutione*[(*)] and thereto enclosed in Hudden's Analysis: If, however, all roots are possible, he believes, they are not permitted [habe nicht statt], as is shown in the case of solving the division of an angle in three parts. The formula can be derived in various ways; I have (in 699) followed Herrn Euler's in his *coniectatione de formis radicum aeqationum Comm. Ac. Petrop. T. VI.* p. 216. §. 3;4[3].

718. Scholium. This apparent inconsistency [Ungereimtheit] (708) has given mathematical understanding many headaches. Newton says in *Ar. uniu. de Aequat. reductione per diuisores surdos* Page 209 of Gravesand's[4] Edition: "If it [the equation] gives three possible roots, then they all act [verhielten sich] indifferent to the coefficients of the equation, and are expressed by *x* without any difference, hence none are found due to the fact that all three expressed according to this law are not possible." It is easily seen that this says nothing: for even therefrom the same manifold of the roots arises, such that various unknown magnitudes, contrary to the given ones, through which the coefficients become determined, have all the same manner, whether these unknown magnitudes are all possible, or a couple of these are impossible, just as example 242 demonstrates. Meanwhile, this position of Newton's prompted Friedrich Wilhelm Stübnern to find here an application of the Principle of Sufficient Reason, since there was no existing reason as to why any one of the three roots should present itself more readily than any of the others (in his 1733 disputation held at Leipzig *contra virium mensuram cartesianam pro leibnitiana* § 134). Herr Clairaut (in Mylius's German translation of his *Algebra*, Parts V. VI. and further § 302. S.) also treated this formula and noticed that both the impossible cubic roots could, perhaps, still yield a possible sum, since the impossible magnitudes could cancel themselves out. He proceeds to transform each of the impossible cubic roots into an infinite series according Nicole's instruction (Mem. de l'Ac. des Sc. 1738, p 99-100), and

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[1] Niccolò Fontana Tartaglia (1500 – 1557)

[2] Rene Descartes (1596 - 1650)

(*) In the Addendum to his *Commentar. in Geom. Cartes.* in op cit. *Aufl. 345. S.* as well as the Work *de organica sect. con. descriptione* Leyden [Lugdunum Batavorum] 1646.

[3] This is available in full in Latin thanks to the Euler Archive at: http://www.math.dartmouth.edu/~euler/docs/originals/E030.pdf

[4] The Dutchman Wilhelm Jacob Gravesande was a major promoter of Newtonian philosophy at the beginning of the 18th Century.

shows that the impossible part, which is to be arranged separately in each, will cancel by addition. This uncommonly laborious procedure is entirely superfluous, if the matter is treated according to 702; 709, for them alone, however, some doubts remain.

719. Scholium. For it does not follow, that a magnitude is possible, just because it can be expressed by a series in which all the terms that might be found are possible. For an increasing series can be found for $\sqrt{1-u}$ (625). This series' terms, in so far as they can be found, are all possible, regardless of whether the magnitude of **u** may be smaller or larger than 1. However, in the latter case, the series expresses an imaginary number, and yet it expresses such through purely possible terms. It is true that the series does not, thereupon, approach the value of the root, which it should express, but rather diverges from it (12), however, this does not demonstrate that it expresses something impossible. For the series in 13, for example, expresses the possible value of $\frac{1}{1+x}$; where x can be smaller or larger than 1, and yet, in the latter cases, it diverges from the true values. If this very series contains purely possible terms, and yet signifies some impossible, then its supplement [Ergänzung] (11) will be impossible. Although, thus, the series, which according to the procedure of Nicole is the sum of two series with possible terms, contains purely possible terms, yet it still must be investigated how it appears with the supplements [Ergänzungen], and whether, were the terms impossible, would also cancel themselves out.

720. Scholium. Incidentally, Leibniz had already in 1696 (see his Letter to Wallis *Wallis. Opera Vol. III, coll. letters, letter 27*) made an observation [Erinnerung] just like that of Nicole, namely, that the impossible magnitudes in the sum, as he says, *virtualiter* cancel themselves out; thus, Nicole mistakenly boasts to have been the first to see this. Konig (*Mem. de l'Ac. de Prusse 1749*, p.180) regarding this appearance of an impossibility, which he held to be something real, wished to account for it entirely with Logic, just as Stübner explained it with Metaphysics, such that, thus there only lacked a philosopher who would call to Ethics for help. König shows, namely, that the common analysis, through which the formula is found, presupposes something, which leads to an impossibility if all three roots are possible. He thus hopes, that an improvement of the analysis would disclose a formula, whereby this inconsistency [Ungereimtheit] would not occur. However, Harriot, Wallis (*Algebra 45;46; Cap. Op. Vol. II*), Euler, Landen (*Mathem. Lucubr. P. VI*) have each arrived at just the same formula by a suitable analysis, and shed light upon 716, such that nothing else can be brought out.

721. Corollary. In the case of (710), *m* and *n* (699) can be found through an approximation.

Example. $x^3 = 6x - 40$; Here $a = 6$; $b = -40$; thus, $c^2 = 392$, and x = $\sqrt[3]{-20+\sqrt{392}} + \sqrt[3]{-20-\sqrt{392}}$ where the cubic root is extracted by approximation after beginning with the square root, and thus a nearly true value of x could be found.

720. Corollary. Were, however, the actual value of x rational, then it would not be found with this procedure.

For example, 392 = 98 × 4 = 14 × 7 × 4 = 2 × 7 × 7 × 4, thus $\sqrt{392}$ = $14\sqrt{2}$. Now, however, $(-2\pm\sqrt{2})^3$ = $-20\pm14\sqrt{2}$ as can be affirmed through the trial, save that I must necessarily show here how it will be found. Thus, x = $-2+\sqrt{2}-\sqrt{2}$ = -4.

721. Corollary. In 708, however, where the approximation (719) could not be applied, there must be a method, provided the formula should be practical, to extract the cubic root out of $K\pm\lambda$ (702), or to find P and Q (702). The method employed in 702 onward, will not serve to this end, since it directs to find P out of an equation, which is no more simpler than the last, except that the root is to be found through Cardan's Formula (712).

722. Corollary. A way analogous to the procedure in 702 could be sought in which the cubic root is picked out of a two-part magnitude, with one part rational, the other irrational, yet still being possible. This cubic root must assume an expression, which consists of a rational part and an irrational part, cubing both sides, and setting the rational equal to the rational, and also the irrational equal to the irrational. However, then also both parts of the adopted expression, which are sought, are still left to be found through a cubic equation, for whose solution Cardan's Formula must be employed for such calculation. Thus, Wolf, *El. Analysis* § 360, to find the cubic root of $20+\sqrt{392}$, solves the equation $z^3-6z=40$, for the value of *z*, which he finds is 6 by some trials, and thereupon he finds the sought cubic root $2+\sqrt{2}$. With this cubic root, he hence wishes to know if it is used to solve the equation $x^3-6x=40$, just as Cardan's Formula gives it. Could he not have directly found the value of *x* in this equation just as easily?

723. Scholium. The analysts have sought methods to extract roots of every degree out of a two-part magnitude, whose one part is rational, the other irrational, for here cubic roots are especially necessary. Newton, *Ar. uniu. de reductione radicalium ad simpliciores; cet. p. 50*, has provided instructions respecting such, although without proof, therewith various mathematical minds have employed themselves, including *Colin MacLaurin, Algebra P. I.* § 118, and further §. Clairaut *Alg. IIII. Th. XXV.*, among other works. These treatments demanded, however, so much extensiveness that, on account of this, various authors, even Agnesi in her rather complete Introduction to Algebra, have omitted them. Attempts are still frequently provided, as with Colin MacLaurin *Alg. P. I. 131.* §, which performs what is here regarded to be accomplished by 721, satisfying itself to designate a path, which is not wholly based upon experiments. Before the manner of help which the cases (708; 721) provided was known, it had been called *casum irreducibilem*, which according to Leibniz's report (epistle to Oldenburg; in the Collected Epistles of Wallis, Opera Vol. II) Raphael Bomelli was the first who taught how to find the roots of this seemingly impossible magnitude. They will excuse me, if I therefore do not completely furnish them here. I would have omitted Cardan's Formula, which according to the credentials of much mathematical understanding, including that of Agnesi, op cit. 180 §, is of very little use, were the numerous and peculiar efforts which the Algebraicists have exerted upon a difficulty which is non-existent, not remarkable. Perhaps, after we have convenient methods to find the rational and irrational roots, their theory, which they themselves would not employ for actual application, just as I have reported it here, also will not be disagreeable. Landen, op cit., still has taken on the effort to find such a formula for the cubic equation where the second term is not absent, whereupon I must judge, the same as in (513). Similar formulas for the biquadratic equation have been sought, and Herr Euler has established a conjecture based upon this [Cardan's] Formula, as well as the root of the general equation appearing surmountable (see 717). I have thoroughly examined the preceding considerations in a Programme[5] issued here in 1757, *Cardani sequationum cubicarum radices omnes tenere cet*.

[5] Cassells New German Dictionary says: "Programme: annual report (in German schools) published at Easter, usually containing some scientific essay by one of the teachers."