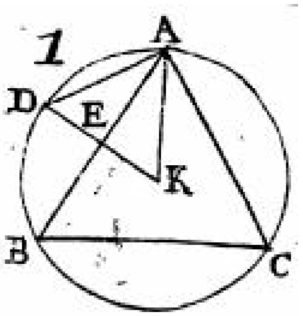
Anfangsgründe: Analysis of Finite Magnitudes
von Kästner

Geometrical Problems

153. Problem: To find the Proportion of the Side **AB** of an equilateral Triangle **ABC** (*Fig. 1*) to the Radius **KA** of the Circle in which it is described.



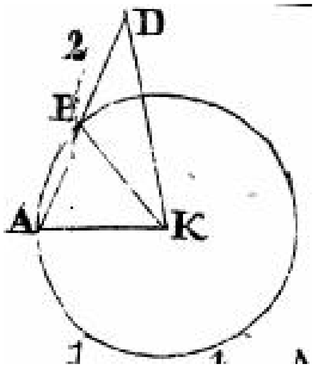
Solution: The Arc **ADB** is 120°. (*G. 23. S. 2 Z.*). Dropping **KED** perpendicular upon **AB**, then $DKA = 60°$, and $AE = \frac{1}{2} AB$ (*G. 22. S. 2 Z.*). Also $AD = AK = DK$ (*G. 23. S. 5 Z.*), and consequently, $DE = EK$ (*G. 9. S. 5 Z.*) $= \frac{1}{2} AD$. Therefore, $AE^2 = AD^2 - DE^2$ (*G. 15. S*). Setting the Radius = r, the sought Side = x, then $AE = \frac{1}{2} x$ and $AE^2 = \frac{1}{4} x^2 = r^2 - \frac{1}{4} r^2 = \frac{3}{4} r^2$ or $x^2 = 3r^2$. Therefore, $x = \sqrt{(3r^2)} = r\sqrt{3} = r \,.\, 1.7320508$ (*Ar. IIII. 27*).

154. Corollary: $AE = 0.8660254$. Moreover, $\sin 90° : \sin 60° = r : AE$ (*Trig. 2 Def.*) or $AE = (r \,.\, \sin 60°) / \sin 90°$. Thus, supposing **r** has as many Parts as the whole Sine of the Table, then **AE** becomes the Sine of 60° as stated in the Table. For Example, 8660254 here, if $r = 10\,000\,000$.

155. Remark: The Side of the Polygon of 2**n** Sides, **z**, is found from the Side of the Polygon of **n** sides, **f**, (*G. 43. S*), by $z^2 = 2r^2 - 2r\sqrt{(r^2 - \frac{1}{4} f^2)}$. Consequently, $2r \,.\, \sqrt{(r^2 - \frac{1}{4} f^2)} = 2r^2 - z^2$ and upon taking the sides squared $4r^4 - r^2f^2 = 4r^4 - 4r^2z^2 + z^4$ or $4r^2z^2 - z^4 = r^2f^2$ or $(z/f) \,.\, \sqrt{(4r^2 - z^2)} = f$, which then serves to find the Side of the Polygon of 2**n** Sides from the Side of the Polygon of **n** Sides.

Were one to set $z = r$, then from out the same **f**, one would find the Side of the Triangle.

156. Problem: To find the Proportion of the Side of the Decagon $AB = x$ (*Fig. 2*) to the radius $AK = r$.



Solution: Were one to extend the side [**AB**], then, if **KD** is drawn at will, $BKD + BDK = ABK$ (*G. 13. S. I. Z*). Now, $AKB = 4R/10 = 2R/5$, therefore, $ABK = 4R/5$ (*G. 13. S. 5. Z*) $= 2AKB$.[*] Therefore, taking $BD = BK$, then $BKD = BDK = AKB$ and, due to the mutual angle at **A**, $\Delta\, ABK \propto \Delta\, AKD$ (*G. 27. S*). (I name the vertices of the triangles in the same order in which each equal angle stands in its triangle. For example, the angle **K** in the triangle **ABK** is equal to the angle **D** in the triangle **AKD**.) Therefore, $AK:AB = AD:AK$ or, $r:x = r + x:r$ or $r^2 = x^2 + rx$ or $x^2 = r^2 - rx$. Instead of the aforementioned **z**, **p**, **q**, in (78), here **x**, **-r**, $+ r^2$, are substituted, giving $x = (-1/2).r$

---

[*] [Trans. note] Kästner defines a right angle [rechte Winkel] in (*G. 11 Erkl.*). Then, in the propositions, he employs the symbol '**R**' to denote a right angle (*G. 6. S. 2. Anm.*).

$\pm\sqrt{\frac{1}{4}\mathbf{r}^2+\mathbf{r}^2}$. Here, the negative sign yields something negative for the root, and yet, one regards without doubt the positive [root] as the sought side. Therefore, only the positive sign is employed, and this gives $\mathbf{x} = \sqrt{\frac{5}{4}\mathbf{r}^2} - \frac{1}{2}\mathbf{r}$.

157. <u>Prepatory Drawing [Verzeichnung] (constructio geometrica.)</u>
Drawing **CG = r** perpendicular to **CF = ½ r**, and describing about **F** the arc **GH** with **FG** = $\sqrt{\frac{5}{4}\mathbf{r}^2}$, (*G. 15. S*), then **CH = FH.FC = x**.

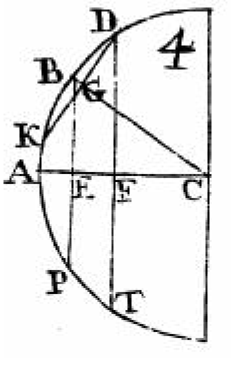
158. <u>Corollary</u>: While $\mathbf{x} = \frac{\mathbf{r}\sqrt{5}}{2} - \frac{1}{2}\mathbf{r} = \mathbf{r}\frac{\sqrt{5}-1}{2}$, then sin 18° = $\mathbf{r}\frac{\sqrt{5}-1}{4}$.

159. <u>Corollary</u>: If, for the side of the Pentagon, as in (*155*), **z = x** (*156*) $= \mathbf{r}\frac{\sqrt{5}-1}{2}$, then $4\mathbf{r}^2 - \mathbf{z}^2 = 4\mathbf{r}^2 - \frac{5-2\sqrt{5}+1}{4}\mathbf{r}^2 = \frac{\mathbf{r}^2(5+\sqrt{5})}{2} = \frac{\mathbf{r}^2(\sqrt{5}+1)\sqrt{5}}{2}$. Consequently, this side of the Pentagon $\mathbf{f} = \frac{\mathbf{r}(\sqrt{5}-1)}{2} . \frac{\mathbf{r}\sqrt{(\sqrt{5}+1)\sqrt{5}}}{2\mathbf{r}} = \frac{\mathbf{r}\sqrt{(\sqrt{5}+1)(\sqrt{5}-1)\sqrt{5}}}{4}$ (*28*) $= \frac{\mathbf{r}\sqrt{(5-1)\sqrt{5}}}{4}$ (*Ar. II. 5*) $= \frac{\mathbf{r}\sqrt{5}}{2}$ (*24*) $= \mathbf{x} + \frac{\mathbf{r}}{2}$ = **FH** (*157*).

160. <u>Problem</u>: To find the side of the Octagon in a given circle.

<u>Solution</u>: The side of the Square is $\mathbf{r}\sqrt{2} = \mathbf{f}$ (*155*), therefore, as in that place $\mathbf{z}^2 = 2\mathbf{r}^2 - 2\mathbf{r}\sqrt{\mathbf{r}^2 - \frac{1}{2}\mathbf{r}^2} = 2\mathbf{r}^2(1-\sqrt{\frac{1}{2}})$.

161. <u>Corollary</u>: Moreover, by (*153*, *155*, *156*, *159*, *160*), the polygons of 3; 6; 12; 24; etc., of 4; 8; 16; 32; etc., of 5; 10; 20; etc. sides can be constructed in the circle, and their sides be measured by the radius.



162. <u>Corollary</u>: Thence, the sines of half the center angles [Mittelpunctswinkel (*G. 23. S. 3. Z*)] of these Polygons is produced, in the same manner as in (*154*).

163. <u>Problem</u>: If **BP = k**, **DT = c**, the parallel chords of the two arcs **BAP**, **DAT** [*Fig. 4*], are given, find, from there, the chord of their difference, which shall be called **h**.

<u>Solution</u>: **BE = ½ k**, **DF = ½ c** are the respective sines of half of the arcs **BAP**, **DAT**, and **DG** is the sine of their difference. Between these 3 sines, one finds the comparison from *Trig. 29 Theorem*. Now, **DAT – BAP = BD + PT = 2BD**, and since the difference of the arcs, to which the chords appertain, once again then great as the difference of the Arcs, to which the sine appertains, consequently, **DG** the sine of the half difference of the arc, to which the chord appertains, and therefore, **ZDG = h**. Therefore, setting in the places in cited *Trigonometry*, **s = ½ c, m = ½ k, d = ½ h**, then **½ h =**

$$\frac{\frac{1}{2}\mathbf{c}\sqrt{\mathbf{r}^2-\frac{1}{4}\mathbf{k}^2}-\frac{1}{2}\mathbf{k}\sqrt{\mathbf{r}^2-\frac{1}{4}\mathbf{c}^2}}{\mathbf{r}}\text{, therefore, }\mathbf{h}=\frac{\mathbf{c}\sqrt{4\mathbf{r}^2-\mathbf{k}^2}-\mathbf{k}\sqrt{4\mathbf{r}^2-\mathbf{c}^2}}{2\mathbf{r}}.$$

164. <u>Corollary</u>: If, in *Fig. 5*, the arc **HλL = ½ HLM**, then in (*163*) one sets **h = k = HL**, **c = HM**. One sees, namely, the chord **HAL** as the difference of the chord **HLM** and of one, which is as great as **HλL**. If **r** = 1, this gives (*Trig. 19, S. Anm.*)

$$2\mathbf{h}=\frac{\mathbf{c}\sqrt{4-\mathbf{h}^2}-\mathbf{h}\sqrt{4-\mathbf{c}^2}}{2\mathbf{r}},\quad\text{therefore,}\quad\mathbf{h}(2+\sqrt{4-\mathbf{c}^2})=$$

$\mathbf{c}\sqrt{4-\mathbf{h}^2}$. If one squares the sides, then one finds finally, $\mathbf{h}^2=\frac{\mathbf{c}^2}{2+\sqrt{4-\mathbf{c}^2}}$, for **h**, the chord of the half arc, if **c** is given as the whole chord.

166. <u>Corollary</u>: Since the double arc is the sum of two single arcs, then in (*164*) one sets **h = k**, **r** = 1. This yields $\mathbf{c}=\mathbf{h}\sqrt{4-\mathbf{h}^2}$ as the chord of the double arc, by way of the chord of the single arc.

167. <u>Corollary</u>: One seeks **h** in (*166*), that is, $\mathbf{c}^2=4\mathbf{h}^2-\mathbf{h}^4$ or $\mathbf{h}^4=4\mathbf{h}^2-\mathbf{c}^2$, then, consequently, (*78*), $\mathbf{h}^2=2\pm\sqrt{4-\mathbf{c}^2}$. This gives two different values for the square of the chord **h** of the single arc from the chord **c** of the double arc. That is, the chord **HM** (*Fig. 5*) appertains equally to the arcs **HLM** and **HNM**. It follows that halving the arc whose chord is **HM**, denotes each of the named arcs halved. If, however, **HLM = ω,** the circumference **HLMNH = P**, then both arcs, to which the chord **HM** appertains to are **ω; P – ω**. Therefore, their halves **HλL = ½ ω, HηN = ½ P - ½ ω**, which, together constitute ½ **P**. Consequently, both the values of $\mathbf{h}^2$ appertain to both the squares of the chords **HL, HN**. If now **ω < ½ P**, then ½ **ω** < ¼ **P.** However, ½ **P** – ½ **ω** > ¼ **P** that is, the chords **HL < HN.** In the case then, if the smallest arc, to which the chord **c** appertains, is smaller than a quadrant, then the square of the chord of the half of the arc = $2-\sqrt{4-\mathbf{c}^2}$ and the square of the chord of the half of its supplement to **P** [or, remainder from **P**], = $2+\sqrt{4-\mathbf{c}^2}$**.** In the opposite instance, the case is turned around.

168. <u>Remark</u>: This ambiguity, in virtue of which **h** can signify both **HL** as well as **HW**, became eliminated in appearance in (165), it is, however, also to be found in the action

there, since one can regard the radical quantity in the divisor as positive as well as negative.

169. Remark: The values from (*165;167*) are also both equal, namely, if one multiplies the divisor and dividend of (*165*) with $2-\sqrt{4-\mathbf{c}^2}$, then the magnitude remains unchanged, and its expression transforms itself into the expression in (*167*).

170. Remark: The ambiguities are to be found just the same as one seeks the description of the sine or tangent of the half arc by (*Trig. 19. S.*). (*167*) I have treated this generally for every division of the angle in my report: *Unde plures insint radices æquationibus sectiones angulerum definientibus*, Gott. 1756.

171. Remark: Through the repetition of the task (*167*), one calculates the chords [Sehnen] for ¼ **ω;** $\frac{1}{8}$ **ω;** etc., or for all divisions, which might be found by halving, that the Algebra has here so far followed the common Geometry (*G. 5. S. Z*).

172. Corollary: If one had the sides of two regular polygons of **m** and **n** angles, that is, the chords of [the arcs] (1/**m**)**P** and (1/**n**)**P,** then one finds, by (*163*), the chord of $\frac{\mathbf{m}-\mathbf{n}}{\mathbf{mn}}\mathbf{P}$. Were **m – n** now a power of 2, then one finds, by (171), the chord of $\frac{1}{\mathbf{mn}}\mathbf{P}$, that is, the side of the polygon which has **mn** angles.

Let **m** = 5, **n** = 3. Then one finds, from the sides of the Pentagon and the Triangle (*159; 153*), the chord of [the arc] $\frac{2}{15}\mathbf{P}$, by whose halving the Pentakaedecagon is found. See *Clav. Schol. ad Eucl. IV.16*[*]. If one, as in (*164*), seeks the chord of the sum of $\frac{1}{3}\mathbf{P}+\frac{1}{5}\mathbf{P}$, that is, of $\frac{8}{15}\mathbf{P}$, then one would arrive at the Pentakaedecagon also by three halvings.

173. Remark: Polygons, outside of those which were found in (*161; 171*), do not allow themselves to be found by a solution to a quadratic equation and, moreover, are not registered with common Geometry, as will become more extensively shown below.

174. Problem: From the given side of a regular polygon, to find the radius of the circle which would describe it.

Solution: One need only examine the equation which gives the side of the polygon by the radius, taking the latter as the sought magnitude and the former as the given.

---

[*] The reference is to Christoph Clavius' Commentaries on Euclid (Rome 1589). The Jesuit Clavius was a contemporary astronomer of Kepler, who oversaw the Gregorian calendar reform, involving the question of determining the date of Easter. The referenced passage here can be found in *Opera Mathematica of Christoph Clavius* pg. 159-165. The entire *Opera* is available online at http://mathematics.library.nd.edu/clavius.

E.g., In (*156*), $\frac{2\mathbf{x}}{\sqrt{5}-1} = \mathbf{r}$.

175. <u>Problem</u>: Given radius = 1, with sine = **a** and cosine = **b**, of a single angle, find the sine and cosine of the multiple angle.

<u>Solution</u>: The multiple angle results by the single angle added to itself repeatedly. Therefore, let be in *Trig. 19. S* **d = a, δ = b.** Further, for the double angle, **m = a, μ= b**. Thus, for the double angle, the sine = 2**ab** and the cosine = $\mathbf{b}^2 - \mathbf{a}^2$ (*Trig. 19. S. 2. Z*). Now, setting in the place of **m** and **μ**, the sine and cosine of the double angle, then one has for the triple angle, sine = $\mathbf{a}(\mathbf{b}^2 - \mathbf{a}^2) + \mathbf{b} \cdot 2\mathbf{ab} = 3\mathbf{ab}^2 - \mathbf{a}^3$, and cosine = $(\mathbf{b}^2 - \mathbf{a}^2) \cdot \mathbf{b} - 2\mathbf{ab} \cdot \mathbf{a}$ = $\mathbf{b}^3 - 3\mathbf{a}^2\mathbf{b}$. In turn, using this in the place of **m, μ,** gives, for the quadruple angle, sine = $(\mathbf{b}^3 - 3\mathbf{a}^2\mathbf{b}) \cdot \mathbf{a} + (3\mathbf{ab}^2 - \mathbf{a}^3) \cdot \mathbf{b}$ = $4\mathbf{ab}^3 - 4\mathbf{a}^3\mathbf{b}$, and cosine = $(\mathbf{b}^3 - 3\mathbf{a}^2\mathbf{b}) \cdot \mathbf{b} - (3\mathbf{ab}^2 - \mathbf{a}^3) \cdot \mathbf{a}$ = $\mathbf{b}^4 - 6\mathbf{a}^2\mathbf{b}^2 + \mathbf{a}^4$. This, in place, handed down to **m, μ,** gives the quintuple angle sine = $(4\mathbf{ab}^3 - 4\mathbf{a}^3\mathbf{b}) \cdot \mathbf{b} + (\mathbf{b}^4 - 6\mathbf{a}^2\mathbf{b}^2 + \mathbf{a}^4) \cdot \mathbf{a}$ = $5\mathbf{ab}^4 - 10\mathbf{a}^3\mathbf{b}^2 + \mathbf{a}^5$, and cosine = $(\mathbf{b}^4 - 6\mathbf{a}^2\mathbf{b}^2 + \mathbf{a}^4) \cdot \mathbf{b} - (4\mathbf{ab}^3 - 4\mathbf{a}^3\mathbf{b}) \cdot \mathbf{a}$ = $\mathbf{b}^5 - 10\mathbf{a}^2\mathbf{b}^3 + 5\mathbf{a}^4\mathbf{b}$; and so forth.

176. <u>Corollary</u>: The explicitly calculated sines and cosines observe the following law: If one raises **b + a** to the Power, whose exponent signifies the Multiplicity of the Angle, and without concern for the sign of the terms of this power, but rather only regarding the magnitude yielded, then the sine comprises the second, fourth, sixth, etc. term of the power, and the cosine comprises the first, third, fifth, etc. term of power; the former the even terms, the latter the odd. The signs of the terms, however, alternate in each of the series for the sine and cosine, such that the first term is positive.

The example is provided when one compares the series for the fifth power of **b + a**, which one finds in (*116*), with the sine and cosine of the quintuple angle [as given here].

177. <u>Corollary</u>: One makes $(\mathbf{b} + \mathbf{a})^{\mathbf{n}}$ as in the Binomial Theorem (*141*), namely, by swapping, what is there, **a** and **b**, and placing here **n** in the stead of the **m** there. Further, 4**r** replaces the **n,** that therefore **P** signifies the coefficients whose number is 4**r**, permitting a double division by 2 [that is, division by 4], (numerus partier par). Further are **Q, R, S, T,** the proximate coefficients, there the number for **R**, namely, 4**r + 2**, permits only a single division by 2, and **A, B, C, D,** etc. , are the coefficients at the beginning, namely, **A = n, B =** $\frac{\mathbf{n} \cdot (\mathbf{n} - 1)}{1 \cdot 2}$, **C =** $\frac{\mathbf{n} \cdot (\mathbf{n} - 1) \cdot (\mathbf{n} - 2)}{1 \cdot 2 \cdot 3}$**,** and so on. This then gives the observed law (*176*), when it is generally expressed for the **n**-tuple angle.

Cosine = $\mathbf{b}^{\mathbf{n}} - \mathbf{B}\mathbf{b}^{\mathbf{n}-2}\mathbf{a}^2 + \mathbf{D}\mathbf{b}^{\mathbf{n}-4}\mathbf{a}^4 \ldots + \mathbf{P}\mathbf{b}^{\mathbf{n}-4\mathbf{r}}\mathbf{a}^{4\mathbf{r}} - \mathbf{R}\mathbf{b}^{\mathbf{n}-2\mathbf{r}}\mathbf{a}^{2\mathbf{r}} + \mathbf{a} = \boldsymbol{\mu}$

Sine = $\mathbf{A}\mathbf{b}^{\mathbf{n}-1}\mathbf{a} - \mathbf{C}\mathbf{b}^{\mathbf{n}-3}\mathbf{a}^3 + \mathbf{E}\mathbf{b}^{\mathbf{n}-5}\mathbf{a}^5 \ldots + \mathbf{Q}\mathbf{b}^{\mathbf{n}-4\mathbf{r}-1}\mathbf{a}^{4\mathbf{r}+1} - \mathbf{S}\mathbf{b}^{\mathbf{n}-4\mathbf{r}-3}\mathbf{a}^{4\mathbf{r}+3} \ldots = \mathbf{m}$

Thence, it follows for the (**n** + 1)-tuple angle that **bμ – am =**
Cosine = $\mathbf{b}^{\mathbf{n}+1} - (\mathbf{A}+\mathbf{B})\mathbf{b}^{\mathbf{n}-1}\mathbf{a}^{2} + (\mathbf{C}+\mathbf{D})\mathbf{b}^{\mathbf{n}-3}\mathbf{a}^{4} \ldots$
$-(\mathbf{Q}+\mathbf{R})\mathbf{b}^{\mathbf{n}-4\mathbf{r}-1}\mathbf{a}^{4\mathbf{r}+2} + (\mathbf{S}+\mathbf{T})\mathbf{b}^{\mathbf{n}-4\mathbf{r}-3}\mathbf{a}^{4\mathbf{r}+4} \ldots$

and similarly **aμ + bm** =
Sine = $(\mathbf{A}+1)\mathbf{b}^{\mathbf{n}}\mathbf{a} - (\mathbf{B}+\mathbf{C})\mathbf{b}^{\mathbf{n}-2}\mathbf{a}^{3} \ \ldots + (\mathbf{P}+\mathbf{Q})\mathbf{b}^{\mathbf{n}-4\mathbf{r}}\mathbf{a}^{4\mathbf{r}+1} - (\mathbf{R}+\mathbf{S})\mathbf{b}^{\mathbf{n}-4\mathbf{r}-2}\mathbf{a}^{4\mathbf{r}+3} \ldots$

However, by (*136*), **A + B, C + D, Q + R, S + T,** are the second, fourth, (4**r** + 2)-th, (4**r** + 4)-th coefficients of the (**n** + 1) power, in the same way that **B, D, R, T,** are the second, fourth, (4**r** + 2)-th, (4**r** + 4)-th coefficients of the **n** power. Therefore, when the Law is assumed for the cosine of the **n**-tuple angle, then it follows for the cosine of the (**n** + 1)-tuple angle. In the same way, the sine of the latter follows from the sine of the former, since **A** + 1**, B + C, P + Q, R + S,** and **A, C, Q, S**, are the first, third, (4**r** + 1)-th, (4**r** + 3)-th, coefficients for the (**n** + 1) and **n** powers, respectively. Thus, if the assumed law is true for **n** = 2, 3, 4, 5, then it is true for all multiple angles, concluding in the manner which was used in (*139*).

178. Remark: The Tangents of the multiple angles also permit themselves to be found in this manner, just as in *Trig. 19. S. 3. Zus.* One can determine their general Law, which permits itself to be accomplished still easier by use of the Integral Calculus.

[MMF]