# Sufficient Harmony: An Introduction to the Scientific Method of Carl Friedrich Gauss, as a Continuation of that of Johannes Kepler

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In Chapter 32 of the New Astronomy, Kepler begins his long stretch towards his discovery that the planets move in elliptical, not circular orbits around the Sun. And at first glance, his method for making this discovery might appear to be confusing, or even to lack rigor. At one point he swaps the arithmetic for the geometric mean, saying that they are "almost equal," and again, later, treats the physical and optical equations, two clearly different angles [see image] as equal, and continues in such a fashion until he ends up with a *larger* error than the one he set out to try to remove. He then declares a battle won, and in fact proceeds to win the war, discovering what is now called "Kepler's first law." Confusion about this method has led some to call it "sleepwalking," or to declare that Kepler made his discovery clumsily, or by accident! But a look at what Kepler's method actually was—and how it is in complete conformity with what Gottfried Wilhelm Leibniz (1646-1716) called the principle of sufficient reason, where a formally "rigorous" mathematical/mechanical treatment would not have been successful—will shed an indispensable light on the method of discovery which underlies the true genius of Carl Friedrich Gauss.

## Sufficient Reason

> The great foundation of mathematics is the principle of contradiction, or identity, that is, that a proposition cannot be true and false at the same time; and that therefore A is A, and cannot be not A. This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles. But in order to proceed from mathematics to natural philosophy, another principle is requisite, as I have observed in my *Theodicy*: I mean, the principle of a **sufficient reason**, viz. that nothing happens without a reason why it should be so, rather than otherwise . . . if there be a balance, in which everything is alike on both sides, and if equal weights are hung on the two ends of that balance, the whole will be at rest . . . because no reason can be given, why one side should weigh down, rather than the other."[1]
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> [T]hat God wills something, without any sufficient reason for his will . . . [is] contrary to the wisdom of God, as if he could operate without acting by reason . . . [however] I maintain that God has the power of choosing, since I ground that power upon the reason of a choice agreeable to his wisdom. And 'tis not this fatality, (which is only the wisest order of providence) but a blind fatality or necessity, void of all wisdom and choice, which we ought to avoid.[2]

[1] Leibniz to Clarke, Second Letter
[2] Leibniz to Clarke, Third Letter

That "nothing happens without a reason why it should be so, rather than otherwise," seems like a simple enough idea to anyone who gives it just a little thought: if something falls, we think, for instance, that we can be sure that we can attribute some cause to its falling. Maybe it slipped; maybe it was pushed; maybe a million particles of air moved around each other in just the right way and a breeze blew it over. Even if we don't know directly what the reason was, we can be assured there *was* a reason. This single fact accounts for the efficacy (and, not incidentally, as we will see below, the name) of human reason. If any one thing in all the world could occur absent a cause, there would be no surety in knowledge, because all knowledge that *is*, is a knowledge of causes.

With this, there are few people who would argue. However, by accepting this we are presented with one most interesting question: why did *anything* ever happen at all? Put perhaps less modestly, the same question might be "what's the *reason* for everything?" We won't pretend to answer that question directly here, but we will answer another question, by analogy, and in so doing touch upon the topic of this entire report: the scientific tradition initiated by Nicholas of Cusa, reified by the work of Johannes Kepler, defended and developed by the ideas of Gottfried Leibniz and Abraham Kästner, and culminating in the successive work of Carl F. Gauss and Bernard Riemann, only to decline sharply thereafter and limp along haltingly to the present day, awaiting its renaissance in the revolutionary activities of Lyndon LaRouche and the LaRouche Youth Movement today. So, to that end, we'll start not with nothing, but rather with an empty page.

Euclid, in his Elements, begins all of geometry with what he calls a "point:" that which has no width, breadth, or depth. The astute reader quickly recognizes that this is nothing other than nothing at all and, as A. G. Kästner says elsewhere in this report, there is no number of *nothings* which can be combined to obtain a *something*.[3] So if we start with Euclid, we don't start with anything at all, which is fine. [animation: Euclid folding up into a plane, and then a line, and then nothing at all.] So, say we start in geometry with nothing; presuming that we must have something (which is indeed a presumption), for what sort of something would there be sufficient reason for its existence? We have a million things to choose from: the square? The triangle? The pentagon? We can add sides to polygons forever without any limit . . . in fact, the triangle, having the least amount of sides, seems to stand out the greatest of all of them. [animation: flipping through a billion polygons until stopping at the triangle] There is something significant about this state of being the least. It would seem that in order to have sufficient reason to be selected out from the vast sea of possibilities, a thing would have to be either the greatest or the least of the entire range of choices—the maximum or the minimum. And the triangle is indeed the least polygon. But, how did we come to speak of polygons? [animation: morphing through a billion other shapes] There are an infinite number of other possible figures to choose from; which of these could be called the least or the greatest of them? For which one, more than for any other, is there sufficient reason for its existence? Recall Leibniz' example of

[3] http://www.wlym.com/~animations/ceres/liona/geometry.pdf

the scale: if none of these stands out more than any other, none of them will be chosen at all. The largest and the smallest shape are clearly absurdities—nothing can be imagined so small or so large that a thing could not be imagined smaller or larger. [animation: a shape oscillating in size from small to large and back and then to ridiculously small, and then again immensely large] The single thing that all of these shapes have in common is that they have an inside, and a line or lines which contain it. Maybe the answer to our question is neither the least nor the greatest . . . but both. For any given shape, a shape can be imagined which has more contents and less circumference. [animation: Steiner proof] But there *is* a limit to this process: what figure has the greatest area for the least circumference? Look at the circle drawn to your right—can you adjust the perimeter to make the area any greater?

What's more: every figure has circular motion as its first, implicit action. [animation: geometric figures' rotation.] This action is a simple physical expression of the geometric property which distinguishes the circle: the ability to accomplish the most with the least. All translational action can then be derived from circular action acting on circular action. [animation: straight line cycloid] [animation: Peaucellier linkage]

It can be seen with little effort that the entirety of Euclidean geometry can now, in fact, begin to be constructed by the circle, or circular action: that is, by ruler and compass, once circular action acting on circular action has given you the line. [animation: circle creating line and point] [animation: ruler and compass construction] [4]

But now, the introduction of multiple figures gives rise to another type of magnitude that must precede them: a proportion must exist between two similar figures in order for them to actually be different (if there is no proportion between them, they will be the same figure). [animation: similar figures] In fact, for any triangle, a definite proportion must exist between the sides in order for them to be unequal. [animation: polygon breaking up into triangles of various proportional relationships] If we are to divide our line into a given proportion, which division would be favored by sufficient reason as the first? Are any of them more or less arbitrary than the others? Here, it is clear, there is again no maximum division. The minimum, then, would have to be two, since the next smallest number, one, would be no division at all. Also, it seems clear that, so long as our division is into two equal parts, nothing unnatural is being introduced, as a division into one and one is simply two of what we began with.

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But from where did we get our two? Or, better: where did we get our 1:2? In order for us to

[4] There *are* physical curves which cannot be constructed with ruler and compass, but all of these curves are demonstrably produced by circular action acting infinitely on circular action. This is almost certainly the proper interpretation of the results of French physicist and mathematician Jean Baptiste Joseph Fourier (1768-1830) and the development of those results by Pierre Gustav Lejeune-Dirichlet (1805-1859), culminating in the work of Bernhard Riemann (1826-1866). They demonstrated, successively, through their work on physical potential fields, that all mathematical functions which occur in nature can be represented by infinite series of trigonometric (circular) functions. That is, the exemplar of every physical process can be decomposed into circular action acting on circular action.

divide a line in half, the *idea* of one half would have to precede our division.

But maybe that is unavoidable. After all, what kind of division of a line *doesn't* require that the size and number of pieces be known, arbitrarily, in advance? [animation: cycling through the different proportions: 1:2:3, 1:3:4, 5:2:7, etc.] After all, a line can't simply be in some proportion with *itself*, can it? And if it were possible for that to occur, where would that cut be?

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The cut would have to be simultaneously the maximum and the minimum: in this case, both the extremes and the mean of a single ratio. Let's assume that we have such a cut, similar to itself, where, if the entirety, the maximum, is one, the smallest piece, the minimum, is chosen so that the remainder is the mean between the two, forming a constant proportion with one, or itself. [insert pedagogical on approximation of golden mean, proceeding upwards from the ratio 1:2]

In this way, we can begin to account for all things which partake in quantity and proportion, using geometry and number as symbols for their exemplars. As Nicholas of Cusa[5] states in his "On Conjectures:"

> The natural, sprouting origin of the rational art is number; indeed, beings which possess no intellect, such as animals, do not count. Number is nothing other than unfolded rationality. So much, indeed, is number shown to be the beginning of those things which are attained by rationality, that with its sublation, nothing remains at all, as is proven by rationality. And if rationality unfolds number and employs it in constituting conjectures, that is nothing other than if rationality employs itself and forms everything in its highest natural similitude, just as God, as infinite mind, in His coeternal Word imparts being to things. There cannot be anything prior to number, for everything other affirms that it necessarily existed from it.

Now, you will recall our double meaning for the word "reason:" Reason is the word for both a cause, and that which looks into causes. The reason for this should be obvious from the geometry: if human reason is capable of measuring every cause, then every cause must share some similarity to human reason, though differing in proportion, because two things can only measure each other insofar as they are similar, and one is contained proportionally in the other.[6] This is often called Plato's doctrine of reminiscence, because it was validated in a rigorous demonstration which Socrates performed in the *Meno* dialogue.

[5] For Cusa's influence on the work of Kepler and Kästner, see "In Praise of Astronomy" and Kästner's review of Cusa's mathematical works in this report.

[6] This is clear from the fact that the word "reason" is also derived from the word "ratio," hence "rational." It is also for this reason that irrational magnitudes should be considered as misnamed, because even transcendental quantities are rational in the original, broader sense.

"Now Plato's view on mathematical things was that the human mind is in itself thoroughly informed on species or figures, and axioms and conclusions about things. However, when it seems to learn, it is merely being reminded by sensible diagrams of those things which it knows on its own account. He conveys that with singular ingenuity in the Dialogues by introducing a slave who when questioned by his master makes all the replies as desired."[7]

"[N]othing enters into our minds from without, and it is a bad habit we have of thinking as if our soul received certain "species" as messengers and as if it had doors and windows. We have all these forms in our own minds, and even from eternity, for at every moment the mind expresses all its future thought and already thinks confusedly of everything of which it will ever think distinctly . . . This Plato excellently recognized in proposing his doctrine of reminiscence . . ."[8]

Before we can elaborate further on that, however, we have to take note that we passed over something which was introduced earlier, at the moment we began to compound our circular motions. Just as proportion was required in order to have multiplicity of objects, position in space, whiteness, blackness, and difference more generally; something similar is required in order to have a world with more than a single motion.

Every set of motions is combined in some definite proportion. [animation: machinery resembling straight line machine] These proportions can be heard in a simple way in contrasted rhythms [sound: rhythmic juxtaposition, maybe from the same machine] but more profoundly in the motion of a vibrating string. [animation: tones on a vibrating string, with a sliding fret changing the tone. String vibration should be seen and heard.] And just as geometry and arithmetic deal in the exemplars which produce quantity and shape, music is the science which deals solely in the exemplar of harmony:

Music has nothing but the harmonies to keep in view, and seeks for nothing beyond it: it is directed to the sole aim of giving delight.[9]

However,

[T]he philosophers commonly look for harmonies nowhere else but in melody, and . . . for many people it is an unexpected treat when they are told that sounds are something different from the harmonies that are thought to be in sounds . . .

For sensible harmony, or things which are analogous to it, is one thing,

[7] Kepler, *Harmonices Mundi*, Book IV
[8] Leibniz, *Discourse on Metaphysics*
[9] Johannes Kepler, *Harmonices Mundi*, Book IV

harmony which is apart from and purified of sensible things is another. The former are many, both in respect of their subjects, which are different in kind, and individually: but genuine harmony which is apart from sensible subjects is one and the same in whatever kind.

Now our earlier example from the Meno can be made even clearer: the rational soul responds to quantity, proportion, and sufficient reason in motion, in the form of musical harmonies generated from the motion of vibrating strings. Thus, Kepler can say that ". . . harmony . . . is in no way outside the soul, as was made clear above in the example of number"—following Socrates, who says in the Timaeus that "harmony, which has motions akin to the revolutions of our souls, is . . . meant to correct any discord which may have arisen in the courses of the soul, and to be our ally in bringing her to harmony and agreement with herself . . ."

But now, different proportions of the string produce different consonances and dissonances with each other. [animation: two sliding frets on strings stopping at various proportional lengths] Which of these is primary?

Kepler describes seven divisions of the string, and only seven, as having the same "harmonic" characteristic of self similarity as our extreme and mean ratio from above. If, instead of a line, we take a vibrating string as our One, which divisions of the string will give us tonal consonances such that each part is in consonance with the other, and both are consonant with the whole? This is another expression of our mean and extreme ratio—sufficient reason, but with regard to harmonic states. [animation: seven strings sounding three tones each, for every harmonic division] Aside from the properties of self similarity and simultaneous maximality and minimality which sufficient reason demands, these seven are limited by two important factors: constructibility by means of circular action acting on circular action, and the judgment of the soul which was composed in accordance with these harmonies.[10]

From these seven harmonic divisions, the entirety of the musical scale can be built up—or rather, built down, because as can be seen here, the smaller divisions of whole and half steps are constructed by means of intersection of the larger, harmonic ratios. [animation: construction of the musical scale by intersecting harmonies] This is a necessary consequence of sufficient reason: remember Euclid: Euclid's points, planes and lines, the supposed building blocks of geometry, only exist in actuality as the intersection of solid bodies. In all cases, sufficient reason demands that the part be composed of the whole, rather than vice-versa. This becomes the basis of a refutation of the supposition that matter is composed "from the bottom up" by the atoms and particles which are found in it, as we shall see below in the work of Kepler and Gauss.

**A Rather Long Induction**

Kepler's investigation in Chapter 32 of his *New Astronomy* finds him in the middle of demonstrating, ostensibly, that every planet has an equant, or a point about which it is said to

[10] Book 3, *Harmonice Mundi*: http://www.wlym.com/~animations/harmonies/

move equal angles in equal times, serving to act as a sort of clock, measuring out the time or "mean motion" in proportion to the distance traveled along a planetary orbit. [animation][11] As part of that argument, he presents the following proof that the amount of distance covered by a moving planet in one day varies in proportion to its distance from the Sun. That is, looking at the image, that εω is greater than ψδ by the same proportion that αε is less than αδ. The colors in the text correspond to those in the image, where measurements made with respect to the equant, or point of uniform motion, γ, are in red; those made from the Sun, α, are in blue, and those made from the center of the physical orbit, β, are in green. The symbol ≈ means "almost equal to."

γυ:γδ he says, is ≈ υξ:δψ and γφ:γε ≈ φτ:γω. But γδ:γυ ≈ βδ (which is equal to γυ) :αδ because βδ is the arithmetic mean between γδ and αδ, which is *almost* equal to the geometric mean when two numbers are very close. And, in this case, the entire reason for this investigation is that, as Kepler showed earlier in chapter 31, it is impossible to tell from the observations where the equant of the Earth would be located relative to the center of its orbit, due to their being imperceptibly close. [Kepler quote from 32] It is then easily concluded from those (almost) ratios, that υξ:δψ ≈ αδ:δβ.

He then states again that γπ:γε ≈ φτ:γω but γε:γφ ≈ εβ (which is equal to γφ) :αε, again because εβ is the arithmetic (almost the geometric) mean between γε and αε. Therefore, in the same way as before, it is concluded that now φτ:εω ≈ αε:εβ.

From those two conclusions above, he concludes further that αδ:δβ ≈ δβ (or βε) :εα. Remember, all of this is "almost!" But, it is less almost, he says, because really υξ:δψ>αδ:δβ and εω:φτ<εβ:αε, which errors compensate and make it even more the case that υξ:δψ ≈ εω:φτ.

Therefore, if we want to find the change in speed as the distance from the Sun changes, we need to take the physical motions at aphelion and perihelion, or δψ and εω, as equal. Then we have φτ:εω ≈ εω:ξυ or the same thing, φτ:δψ ≈ δψ:ξυ. Thus the ratio of the times for those equal motions, or (watch the color change here!) $\phi\tau/\xi\upsilon = \varepsilon\omega^2/\xi\upsilon^2 = \delta\psi^2/\xi\upsilon^2 = \gamma\delta^2/\gamma\upsilon^2 = \alpha\varepsilon^2/\beta\varepsilon^2$ = (because $\beta\varepsilon^2 = \beta\delta^2 = \alpha\delta\cdot\alpha\varepsilon$) = $\alpha\varepsilon/\alpha\delta$. What you can see here in the change in colors is what Kepler reveals in the next chapters, and what forms the basis of Gauss' return to a Keplerian/Leibnizian dynamics in opposition to a Newtonian mechanical universe. As Kepler says:

> But indeed, if this very thing which I have just demonstrated *a posteriori* (from the observations) by a rather long induction, if , I say, I had taken this as something to be demonstrated *a priori* (from the worthiness and eminence of the Sun), so that the source of the world's life (which is visible in the motion of the heavens) is the same as the source of the light which forms the adornment of the

[11] For an animated work-through of Kepler's entire *New Astronomy*, see www.wlym.com/~animations/New Astronomy/

> entire machine, and which is also the source of the heat by which everything grows, I think I would deserve an equal hearing.

That is, his investigation was guided by what he knew the truth *had* to be, in the same way as a developed harmonic faculty of the human soul (which has not been destroyed by modern music) knows what *has* to be proper relationship among harmonies. The rest of Kepler's investigation is covered in detail on the *New Astronomy* website, but I will cover it here briefly for the sake of comparison.

Kepler knows that there are two suppositions involved in the physics of the vicarious hypothesis which violate the principle of sufficient reason:

1) The change in speed which a planet undergoes cannot take place with reference to a "point" in space because, as we saw earlier with Euclid, points do not exist, particularly not disembodied points. All change in position which is not simply rotational must take place with respect to *something*.

2) If that something is not the center of a circle, then perfect circular action is not possible.

For the first reason given, the existence of an equant is a physical impossibility. So, Kepler sets about seeking a measure equivalent to the equant, but which is *physical* in nature. In this case, that means a cause measured from the Sun, and in accordance with the above demonstrated principles of sufficient reason. He uses his conclusion from chapter 32 to demonstrate that the area swept out by a planet, the measure of the sum of the distances, is roughly proportional to the time as measured by the equant. [image] In the course of doing this he shows that the area which would therefore have to measure the physical equation is roughly equal to the arc subtended by the optical equation. And voila! The physical equation becomes actually physical, and the equant is gone! The fact that his error has now increased does not sway Kepler, because he knows that there is still one more matter to be dealt with before he has set the solar system firmly on a physical footing: that of the circular orbit. Readers who are not familiar with how that is accomplished will enjoy working through the entire process in the pedagogically animated work-through of Kepler's *New Astronomy*, but enough has been said here for us to move on to our main goal.

**Harmony Beneath Discord**

The conditions in which Carl Friedrich Gauss was operating during the period straddling the end of the eighteenth and beginning of the nineteenth century were those of a serious conflict over the nature of the future of the human species. A new nation had just been formed across the ocean, the United States of America, which was the first ever in human history to be based entirely on the principle of republican humanism. The intellectual environment in which Gauss was raised was shaped by vocal supporters and organizers of this revolution, followers of the work of Gottfried Leibniz and Johannes Kepler.[12] But it was also the center of a nightmarish counterattack by the oligarchical feudal interests who were intent on destroying that conception of man and its political expression across the sea, by first destroying any possibility of its taking hold politically in the nations of Europe.[13] As a result, almost the entirety of Gauss' scientific work was accomplished under conditions of occupation. Because of this, Gauss became an expert at appearing to replace the *a priori* methods of Kepler, based on the "worthiness and eminence" of truthfulness of physical principle, with "rather long induction." For Gauss, the target of his eraser was not the equant, but its equivalent: the arbitrary accountant's metrics of Newtonian mass and force.

Two facts would have been known to Gauss by the time Piazzi's observations of the new planet Ceres were made public:

1) Leibniz had already proven, decisively, on the basis of his principle of sufficient reason, that absolute space and time did not exist.[14] This was in explicit contradiction to the Newtonian view which was being peddled through turn of the century Europe by the imperial forces associated

[12] See "Neither Venetians Nor Empiricists Can Handle Discoveries" elsewhere in this report.

[13] See "The Orbit of Gauss" elsewhere in this report.

[14] Leibniz proves this in several locations, but the proof as it appears in his correspondence with Samuel Clarke is the most significant to us here, because Leibniz grounds it solely on the principle of sufficient reason:

"[Newtonians] maintain therefore, that space is a real absolute being. But this involves them in great difficulties . . . I have said more than once, that I hold space to be something merely relative, as time is; that I hold it to be an order of coexistences, as time is an order of successions. For space denotes, in terms of possibility, an order of things which exist at the same time, considered as existing together; without enquiring into their manner of existing. And when many things are seen together, one perceives that order of things among themselves.

"I have many demonstrations, to confute the fancy of those who take space to be a substance, or at least an absolute being. But I shall only use, at the present, one demonstration, which the author here gives me occasion to insist upon. I say then, that if space was an absolute being, something would happen for which it would be impossible there should be a sufficient reason. Which is against my axiom. And I prove it thus. Space is something absolutely uniform; and, without the things placed in it, one point of space does not absolutely differ in any respect whatsoever from another point of space. Now from hence it follows, (supposing space to be something in itself, besides the order of bodies among themselves,) that 'tis impossible there should be a reason why God, preserving the same situations of bodies among themselves, should have placed them in space after one certain particular manner, and not otherwise; why every thing was not placed the quite contrary way, for instance, by changing East into West. But if space is nothing else, but that order or relation; and is nothing at all without bodies, but the possibility of placing them; then those two states, the one such as it now is, the other supposed to be quite the contrary way, would not at all differ from one another. Their difference therefore is only to be found in our chimerical supposition of the reality of space in itself. But in truth the one exactly be the same thing as the other, they being absolutely indiscernible; and consequently there is no room to enquire after a reason of the preference of the one to the other.

"The case is the same with respect to time. Supposing any one should ask, why God did not create everything a year sooner; and the same person should infer from thence, that God has done something, concerning which 'tis not possible there should be a reason, why he did it so , and not otherwise: the answer is, that his inference would be right, if time was any thing distinct from things existing in time. For it would be impossible there should be any reason, why things should be applied to such particular instants, rather than to others, their succession continuing the same. But then the same argument, that instants, consider'd without the things, are nothing at all; and that they consist only in the successive order of things: which order remaining the same, one of the two states, viz. that of a supposed anticipation, would not at all differ, nor could be discerned from, the other which now is."

with the French and British Newtonians and Napoleon Bonaparte.

2) Kepler, in demonstrating this fact earlier, had shown that all matter, space, and time, were not substantial, but accidental [footnote] quantities derived from the harmonies. This was expressed most clearly in his Harmonice Mundi [footnote/link], where he demonstrated that the reason for the spacing and motion of the planets was derived entirely from intersecting harmonic considerations.

In Britain, and in parts of France, Newton's rewrite of Kepler and political burial of Leibniz had taken hold, but within Germany, the tradition of Leibniz and Kepler had been defended by the work of Abraham Kästner.[15] Although Gauss stated explicitly that he would never publicly state his agreement with this view, and although only shadows of it are to be found in his published works, the method of execution utilized in his scientific work make clear his epistemology. The thorough elaboration of these examples will have to wait until the final report from this team currently working out of LaRouche's basement, however a sufficient summary can be presented as an overview.

In Gauss' work on discovering the orbit of Ceres, he doesn't once make use of the Newtonian mass or inverse square law. He briefly mentions that what Newton added to Kepler's laws requires the introduction of the mass of the planet, and the way that the gravitational force generated by that mass affects the Sun. This is because Newton's concepts of mass and force are necessary fictions with respect to each other. Mass can only be determined by observing its response to a force—weighing it on a scale, for example. However a force can only be measured by its effect on mass. The "basic quantities" of Newtonian mechanics are nothing more than a self consistent (up to a point) exercise in circular logic. What's more, none of these quantities are actually applicable to matter, but rather are only applicable to "material points" (such as centers of gravity) which we dispensed with back at Euclid. Kepler, however, derived the properties of the planetary orbits without the aid of either of these fictions. Gauss states—with understandable diplomacy, given the circumstances—in his *Theoria Motus*:

> The laws above stated differ from those discovered by our own KEPLER in no other respect than this, that they are given in a form applicable to all kinds of conic sections, and that the action of the moving body on the sun, on which depends the factor $\sqrt{(1+\mu)}$ , is taken into account. If we regard these laws as phenomena derived from innumerable and indubitable observations, geometry shows what action ought in consequence to be exerted upon bodies moving about the sun, in order that these phenomena may be continually produced. in this way it is found that the action of the sun upon the bodies moving about it is exerted *just as if* an attractive force, the intensity of which is reciprocally proportional to the square of the distance, should urge the bodies toward the center of the sun. If now, on the other hand, we set out with the assumption of such an attractive force, the phenomena are deduced from it as necessary consequences. It is sufficient here merely to have recited these laws, the connection of which with the

[15] Pete's article again? Or maybe we should finally publish Dave Shavin's?

> principle of gravitation it will be the less necessary to dwell upon in this place, since several authors subsequently to the eminent NEWTON have treated this subject, and among them the illustrious LA PLACE [16], in that most perfect work the Mécanique Céleste, in such a manner as to leave nothing further to be desired.[17]

He repeats this sentiment multiple times throughout the course of the book. Again, always with a careful sort of veiled diplomatic delivery, but always making the point for anyone who is willing to listen. This denial of the Newtonian "equants" of mass, force, energy, absolute space and absolute time originates here in his work on astronomy, the first science, but its implications shape the entire body of his work on curvature, potential, and ultimately the hypergeometries of his student Bernhard Riemann.

In a paper, ironically titled *General Propositions relating to Attractive and Repulsive Forces acting in the inverse ratio of the square of the distance*, Gauss eliminates the need for both forces and Newton's inverse square law by redefining the concept of potential as La Place had introduced it in his *Mécanique Céleste*:

> Nature presents to us many phenomena *which we explain by the assumption* of forces exerted by the ultimate particles of substances upon each other, acting in inverse proportion to the squares of their distances apart.[18]

His conceptual underpinnings are often buried underneath pages of "rather long induction" in order to conform to the mind-deadening logical deductive-inductive structure of Euclid's *Elements*, but their core is clear when viewed from the standpoint of Kepler. The only things which can be primary are those conceptions which can be derived immediately from sufficient reason and, as is clear in the above examples of geometry and harmony, that means a universe which is built from the top down, rather than from the bottom up. Matter, then, like the melodic intervals defined by the intersecting harmonies, must be the product of a universe which is unfolding from a single, harmonic, always self-similar whole. This becomes most clear in Gauss' investigation of the secular perturbations of planetary orbits. In the terms of Newtonian astrophysics, the secular perturbations are said to be the effect of gravitating point masses on one another as they pass, deflecting each other from what would otherwise be near perfect elliptical orbits around the Sun. [animation: the goofy gravitating ball one that Liona showed] Could this

[16] The French newtonian Pierre Simon de Laplace was one of Napoleon Bonaparte's mathematics teachers at the *École Militaire*. The first two books of his *Mécanique Céleste*, which Fourier called (with perhaps intentional accuracy) "the *Almagest* of his age," were published in 1799, the same year Gauss launched his first (and last) explicit public attack on the French Newtonians. We present here an excerpt from the opening of this latter day Ptolemy's work, in order to juxtapose his thought process to what we have just gone through:

"A body appears to us to be in motion when it changes its situation relative to a system of bodies which we suppose to be at rest; but as all bodies, even those which seem to be in a state of absolute rest, may be in motion; we conceive a space, boundless, immoveable, and penetrable to matter: it is to the parts of this real or ideal space that we by imagination refer the situation of bodies; and we conceive them to be in motion when they answer successively to different parts of space.

"The nature of that singular modification in consequence of which bodies are transported from one place to another, is, and always will be unknown: we have designated it by the name of force; and we are not able to determine any thing more than its effects, and the laws of its action. The effect of a force acting upon a material point is, if no obstacle opposes, to put it into motion; the direction of the force is the right line which it tends to make the point describe."

[17] Gauss, *Theory of the Motion of the Heavenly Bodies Moving About the Sun in Conic Sections.* (Capitals in the original, but italics added.)

[18] emphasis added

perhaps be a demonstration that Kepler's laws were wrong, and that Newtonian mechanics are necessarily correct? After all, Kepler's laws include no attractive forces, and no masses. They dea only with the spacing, orbital velocities, and elliptical properties of the planetary orbits, as these ar derived from the harmonies. How could Kepler's harmonies account for the apparently mechanistic effect of perturbations experienced in planetary orbits? Let's read the introduction to Gauss' paper *Determination of the Attraction Which a Planet Would Exert Upon a Point at an Arbitrarily Given Location, if its Mass Were Distributed Continuously Along the Entire Orbit, in Proportion to the Time it Takes to Traverse its Individual Parts*[19] :

> The secular changes which the elements of a planetary orbit experience owing to the perturbation of another planet, are independent of the position of the latter in its orbit, and their values are the same whether the perturbing planet follows the elliptical path according to the Keplerian laws or whether its mass is considered to be continuously distributed along its orbit such that the sections of the orbit which are traversed in equal times are also given equal amounts of mass, provided only that the periods of the perturbed and perturbing planets are not commensurable. This elegant theorem can be easily proven from the axioms [*Grundsätzen*] of celestial mechanics, even it has not been expressly stated by anyone before now. Hence the following problem arises, which is worthy of interest as much on its own account as on account of the various artifices which its solution requires: to determine exactly the attraction of a planetary orbit or, better said, of an elliptical ring, on a point at an arbitrarily given location, where the thickness of the ring is infinitely small and variable according to the law just laid out.

He goes on to demonstrate that the effect of perturbation depends entirely upon the parameters of the planets' orbit, where the mass only appears as an *effect* of the amount of time spent by the perturbing and the perturbed planets at given point in their orbits—essentially, the length of the daily arcs dealt with by Kepler in the discussion above. [animation: a planet smearing itself along its orbit, glowing in proportion to the density] But also, this is nothing other than the orbital velocities of the respective planets, all of which, as Kepler demonstrates in Book 5 of his *Harmonice Mundi*,[20] are defined by the minimum and maximum orbital velocities of a planet, which it experiences at aphelion and perihelion. These in turn are defined entirely by the harmonies! Gauss has demonstrated clearly, in the domain of astronomy, what sufficient reason teaches must be true generally—and what Kepler and Leibniz already knew—that matter and its physical properties must be derivative effects drawn from the self-reflexive actions of a single principle of sufficient reason.

Again, as the name implies, the most characteristic property of this sufficient reason is that man's reason is its measure. Man's reason, though diverse in its individual expression in individual human beings, is necessarily made in the image of a single process of sufficient, creative reason. Therefore all of creation reflects a single, creative personality, a single Creator, whom it

[19] http://www.wlym.com/~animations/ceres/sky/GaussPlanetMassDist.pdf
[20] http://www.wlym.com/~animations/harmonies/

is the nature, responsibility, and sole pleasure of man to investigate amidst the harmonies which He has placed inside of us and in the universe which surrounds us.  And because of this, as the very existence of Kepler, Leibniz, Gauss, and Riemann demonstrates, man's mind is the measure of all causes, though confusedly at first, until prompted.  This is the core of the method applied by Cusa, Kepler, Kästner, Gauss and Riemann, and it is the method which a modern renaissance, studying them, is obliged to revive.

> For just as sensible things which we had known beforehand, similarly sensible mathematical things, if they are recognized, therefore, elicit intellectual things which are previously present within, so that the things now in actuality shine forth in the soul which were hidden in it before, as if under a veil of potentiality.[21]

[21] Kepler, *Harmonice Mundi, Book IV*