Posthumous Fragments on the Theory of the Arithmetic-Geometric Mean and the Modulus Function.

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Definition and Convergence of the Algorithm.
[III. 361]

Let a and b be two positive real magnitudes and let $a \geq b$. We form from them, the two sequences:

$$\begin{aligned} &a,\ a_1,\ a_2,\ a_3,\\ &b,\ b_1,\ b_2,\ b_3, \end{aligned} \tag{1}$$

in such a way, that any two corresponding terms are the means of the two preceding, and in fact the terms of the upper series are to be the arithmetic mean, the lower, the geometric mean, i.e.

$$\begin{aligned} a_1 &= \frac{1}{2}(a+b), & b_1 &= \sqrt{ab}\\ a_2 &= \frac{1}{2}(a_1+b_1), & b_2 &= \sqrt{a_1 b_1}\\ a_\nu &= \frac{1}{2}(a_{\nu-1}+b_{\nu-1}), & b_\nu &= \sqrt{a_{\nu-1} b_{\nu-1}} \end{aligned} \tag{2}$$

We presuppose, that square roots (2) will all be taken as positive; then the series continues indefinitely, all its terms are entirely determined and have positive real values. Further, from this directly follows:

1. If $a = b$, all terms of the two series are $= a = b$.
2. However, if $a \neq b$, then it follows

$$(a_1 - b_1)(a_1 + b_1) = \frac{1}{4}(a-b)^2,$$

the inequality:

$$b_1 < a_1,$$

and similarly

$$b_2 < a_2, \quad b_3 < a_3, \ldots..,$$

that is, each terms of the lower series is smaller than the corresponding term of the upper series.

3.

$$a_1 < a, \quad b_1 < b$$
$$a_2 < a_1, \quad b_2 < b_1, ...$$

that is, the upper series continually decreases, the upper increases; hence it becomes clear, that each of the two series have a limiting value a_∞ and b_∞ respectively, which lie between a and b.

4. From the relation

$$\frac{a_1 - b_1}{a - b} = \frac{a - b}{4(a_1 + b_1)} = \frac{a - b}{2(a + b) + 4b_1} < \frac{1}{2}$$

follows

$$a_1 - b_1 < \frac{1}{2}(a - b).$$

Similarly

$$a_2 - b_2 < \tfrac{1}{2}(a_1 - b_1) < \frac{1}{4}(a - b), \cdots$$
$$a_\nu - b_\nu < \frac{1}{2^\nu}(a - b),$$

that is, the magnitudes $a - b$, $a_1 - b_1$, $a_2 - b_2$, form a constantly descending series with the limiting value 0. Thus,

$$a_\infty = b_\infty,$$

that is, the upper and lower series (1) have the same limit, and this is smaller than all terms of the first series, but greater than all the terms of the second.

We refer to this limit as the *Arithmetic-Geometric Mean* (agM) between a and b, written as $M(a, b)$. [1]

2. Backwards extension [Rückwrtsverlängerung] of the Algorithm [III. 362]

We will now extend the algorithm (2) backwards. For that purpose, we note that the roots of the quadratic equation

$$x^2 - 2ax + b^2 = 0 \tag{3}$$

are real and positive as a result of the precondition $a \geq b$, and that a is the arithmetic, b the geometric mean between both these roots. Thus, we set

$$c = \sqrt{a^2 - b^2} \tag{4}$$

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$$a_{-1} = a + c, \quad b_{-1} = a - c, \tag{5}$$

then a_{-1} is to indicate that term of the upper series (1), which precedes the term a, and b_{-1} as that term preceding b of the lower series; through a_{-1}, b_{-1} the algorithm has thus been extended backwards.

We now similarly set

$$\begin{aligned} c_{-1} &= \sqrt{a_{-1}^2 - b_{-1}^2}; \quad a_{-2} = a_{-1} + c_{-1}, \quad b_{-2} = a_{-1} - c_{-1}, \\ c_{-2} &= \sqrt{a_{-2}^2 - b_{-2}^2}; \quad a_{-3} = a_{-2} + c_{-2}, \quad b_{-3} = a_{-2} - c_{-2}, ldots \\ c_{-\nu} &= \sqrt{a_{-\nu}^2 - b_{-\nu}^2}; \quad a_{-\nu-1} = a_{-\nu} + c_{-\nu}, \quad b_{-\nu-1} = a_{-\nu} - c_{\nu}, \end{aligned} \tag{6}$$

a_1, a_2, $a_3, \ldots$ can be regarded as the continuation of the upper series (1) to the left, and similarly b_1, b_2, $b_3, \ldots$ as the corresponding extension of the lower series, so that now two series are thus obtained, which are continued [fortsetz] on both sides ad infinitum:

$$\begin{aligned} &\ldots\ldots a_{-3}, a_{-2}, a_{-1}, a, a_1, a_2, a_3, a_4, \ldots\ldots \\ &\ldots\ldots b_{-3}, b_{-2}, b_{-1}, b, b_1, b_2, b_3, b_4, \ldots\ldots \end{aligned} \tag{7}$$

All terms are real and positive, [and] each term of the upper series is greater than the corresponding terms of the lower. The first series decreases from left to right, the second behaves exactly opposite. On the right, both series have the same limit, but on the left, the upper grows past all limits and the lower strays towards zero, except when $a = b$. That is

$$\begin{aligned} c_{-1}^2 =& a_{-1}^2 - b_{-1}^2 = 4ac > 4c^2 \\ c_{-2}^2 =& a_{-2}^2 - b_{-2}^2 > 4c_{-1}^2 > 16c^2, \ldots \\ & c_{-\nu}^2 > 4^\nu c^2, \end{aligned}$$

from which it is clear, that the series c, c_{-1}, c_{-2}, $c_{-3}, \ldots$ and also accordingly a, a_{-1}, a_{-2}, $a_{-3}, \ldots$ must exceed every limit. Further

$$\frac{b_{-1}}{b} = \frac{b}{a_{-1}} < \frac{b}{a}, \quad \frac{b_{-2}}{b_{-1}} < \frac{b_{-1}}{a_{-1}}, \ldots \quad \frac{b_{-\nu-1}}{b_{-\nu}} < \frac{b_{-\nu}}{a_\nu}$$

$$\lim_{n\to\infty} \frac{b_{-\nu-1}}{b_{-\nu}} = 0,$$

i.e. the limit value of the series b, b_{-1}, b_{-2},...is zero. From which is seen, that the series a_{-1}, a_{-2}, a_{-3},...gradually approaches a geometric progression with the quotient 2.

From the definition of the agM is the following theorem, so evident, that it needs no further elucidation:

The agM between any two corresponding terms of the upper and lower series (7) is the same as that between a and b, i.e.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a, b) = M(a_n, b_n) = M(a_{-n}, b_{-n}). \tag{8}$$

Moreover, the truth of the following relations is immediately realized

$$\begin{aligned} &M(na, nb) = nM(a, b); \\ &M(a, b) = aM(1, \frac{b}{a}) = bM(\frac{a}{b}, 1) = M(b, a). \end{aligned} \tag{9}$$

One sees by a numerical example, how fast the algorithm converges toward the limit value. For example, with the calculation of the terms a_4 and b_4 first differ in the 10th decimal place, a_5 and b_5 in the 23erd place; b_5 is already smaller than 10^{-40}. [2]

3. More progression of the algorithm. Identities.

In the same way, we now form the algorithm which leads to $M(a, c)$, and thus set

$$\begin{aligned} &\overline{a}_1 = \frac{1}{2}(a + c), && c_1 = \sqrt{ac} \\ &\overline{a}_2 = \frac{1}{2}(\overline{a}_1 + \overline{c}_1), && c_1 = \sqrt{\overline{a}_1\overline{c}_1}, ... \\ &\overline{a}_\nu = \frac{1}{2}(\overline{a}_{\nu-1} + \overline{c}_{\nu-1}), && \overline{c}_\nu = \sqrt{\overline{a}_{\nu-1}\overline{c}_{\nu-1}} \end{aligned} \tag{10}$$

we obtain the corresponding backwards extension we obtain from the formulas:

$$\begin{aligned} &\overline{b} = \sqrt{a^2 - c^2} = b, && \overline{a}_{-1} = a + b, && \overline{c}_{-1} = a - b \\ &\overline{b} = \sqrt{\overline{a}_{-1}^2 - \overline{c}_{-1}^2} = b, && \overline{a}_{-2} = \overline{a}_{-1} + \overline{b}_{-1}, && \overline{c}_{-2} = \overline{a}_{-1} - \overline{b}_{-1} \\ &\overline{b}_{-\nu} = \sqrt{\overline{a}_{-\nu-1}^2 - \overline{c}_{-\nu}^2} = b, && \overline{a}_{-\nu-1} = \overline{a}_{-\nu} + \overline{b}_{-\nu}, && \overline{c}_{-\nu} = \overline{a}_{-\nu} - \overline{b}_{-\nu} \end{aligned} \tag{11}$$

and it applies for all n relationships:

$$\lim_{n\to\infty} \overline{a}_n = \lim_{n\to\infty} \overline{c}_n = M(a, c) = M(\overline{a}_n, \overline{c}_n) = M(\overline{a}_{-n}, \overline{c}_{-n}). \tag{12}$$

In expansion of the relationships (6) and (11) we set in general

$$c_\nu = \sqrt{a_\nu^2 - b_\nu^2},\ \overline{b}_\nu = \sqrt{\overline{a}_\nu^2 - \overline{c}_\nu^2} \quad (\nu >=< 0), \tag{13}$$

valid for all positive and negative v, and identify

$$a = a_0 = \overline{a}_0,\ b = b_0 = \overline{b}_0,\ c = c_0 = \overline{c}_0,$$

Then, simple relations exist between all these magnitudes. First of all

$$c_1 = \sqrt{a_1^2 - b_1^2} = \frac{1}{2}(a - b);\ c_2 = \sqrt{a_2^2 - b_2^2} = \frac{1}{2}(a_1 - b_1), \cdots$$

[2] 12.

$$c_\nu = \frac{1}{2}(a_{\nu-1} - b_{\nu-1}) \quad (\nu >=< 0), \tag{14}$$

and this formula indeed, according to (6), also applies for negative v.

Furthermore,

$$c_{-1} = \sqrt{4ac} = 2\sqrt{ac},\ c_{-2} = 2\sqrt{a_{-1}c_{-1}},$$
$$c_\nu = 2\sqrt{a_{\nu+1}c_{\nu+1}}, \quad (\nu < 0)$$
$$c = \sqrt{a^2 - b^2} = 2\sqrt{a_1 c_1},\ c_1 = 2\sqrt{a_2 c_2},$$
$$c_\nu = 2\sqrt{a_{\nu+1}c_{\nu+1}} \quad (\nu >=< 0) \tag{15}$$

Furthermore, from (14) follows

$$\begin{aligned} a &= a_1 + c_1, & b &= a_1 - c_1, \\ a &= a_2 + c_2, & b_1 &= a_2 - c_2, \\ a_\nu &= a_{\nu+1} + c\nu + 1, & b_{\nu+1} &= a_{\nu+1} - c_{\nu+1}, \end{aligned} \tag{16}$$

The last formula is identical with (6) for negative ν. Corresponding relations apply for the algorithm leading to $M(a,c)$, namely:

$$\begin{gathered} \bar{b}_1 = \sqrt{\bar{a}_1^2 - \bar{c}_1^2} = \tfrac{1}{2}\sqrt{(a+c)^2 - 4ac} = \tfrac{1}{2}(a-c); \quad \bar{b}_2 = \tfrac{1}{2}(\bar{a}_1 - \bar{c}_1); \dots \\ \bar{b}_\nu = \tfrac{1}{2}(\bar{a}_{\nu-1} - \bar{c}_{\nu-1}), \qquad (\nu >=< 0) \end{gathered} \tag{17}$$

$$\bar{b}_{-1} = \sqrt{4ab} = 2\sqrt{ab}, \quad \bar{b}_{-2} = 2\sqrt{\bar{a}_{-1}\bar{b}_{-1}}, ...$$
$$\bar{b}_\nu = 2\sqrt{\bar{a}_{\nu+1}\bar{b}_{\nu+1}}, \qquad (\nu < 0)$$

$$\begin{gathered} \bar{b}_1 = \sqrt{a_1^2 - c_1^2} = 2\sqrt{\bar{a}_1\bar{b}_1} = 2\sqrt{\bar{a}_2\bar{b}_2}, ... \\ \bar{b}_\nu = 2\sqrt{\bar{a}_{\nu+1}\bar{b}_{\nu+1}}, \qquad (\nu >=< 0) \end{gathered} \tag{18}$$

it follow thus from (17) and (11)

$$\begin{aligned} a &= \bar{a}_1 + \bar{b}_1, & c &= \bar{a}_1 - \bar{b}_1 \\ \bar{a}_1 &= \bar{a}_2 + \bar{b}_2, & \bar{c}_1 &= \bar{a}_2 - \bar{b}_2 \end{aligned}$$

$$\bar{a}_\nu = \bar{a}_{\nu+1} + \bar{b}_{\nu+1}, \qquad \bar{c}_\nu = \bar{a}_{\nu+1} - \bar{b}_{\nu+1} \qquad (v >=< 0) \tag{19}$$

further, from (15) and (16) arises

$$c_{-1} = 2\bar{c}_1, \qquad a_{-1} = a + c = 2\bar{a}_1,$$
$$c_{-2} = 2\sqrt{a_{-1}c_{-1}} = 4\sqrt{\bar{a}_1\bar{c}_1} = 4\bar{c}_2; \quad a_{-2} = a_{-1} + c_{-1} = 2(\bar{a}_1 + \bar{c}_1) = 4\bar{a}_2,$$

$$c_{-\nu} = 2^{\nu}\bar{c}_{\nu}, \qquad a_{-\nu} = 2^{\nu}\bar{a}_{\nu}, \qquad (\nu > 0), \tag{20}$$

from (18) is obtained in the same manner

$$\bar{b}_{-1} = 2\sqrt{ab} = 2b_1; \qquad \bar{a}_{-1} = a + b = 2a_1,$$
$$\bar{b}_{-2} = 2\sqrt{a_{-1}b_{-1}} = 4\sqrt{\bar{a}_1\bar{b}_1} = 4b_2; \quad \bar{a}_{-2} = \bar{a}_{-1} + \bar{b}_{-1} = 2(a_1 + b_1) = 4a_2,$$

$$\bar{b}_{-\nu} = 2^{\nu}b_{\nu}; \qquad \bar{a}_{-\nu} = 2^{\nu}a_{\nu}, \qquad (\nu > 0), \tag{21}$$

thus that

$$c_{\nu} = \sqrt{a_{\nu}^2 - b_{\nu}^2} = \frac{1}{2^{\nu}}\sqrt{\bar{a}_{\nu}^2 - \bar{b}_{\nu}^2} = \frac{1}{2^{\nu}}\bar{c}_{-\nu}$$
$$\bar{b}_{\nu} = \sqrt{a_{\nu}^2 - \bar{c}_{\nu}^2} = \frac{1}{2^{\nu}}\sqrt{\bar{a}_{\nu}^2 - b_{\nu}^2} = \frac{1}{2^{\nu}}b_{-\nu}$$

the aggregate of (20) and (21) gives:

$$a_{\nu} = 2^{-\nu}a_{\nu}; \quad b_{\nu} = 2^{-\nu}b_{\nu}; \quad c_{\nu} = 2^{-\nu}c_{\nu} \quad (v >=< 0) \tag{22}$$

Lastly we note by still one more relation, which is between the square root composed of the elements of the same algorithm:

$$\sqrt{4a_{\nu+2}} = \sqrt{2(a_{\nu+1} + b_{\nu+1})} = \sqrt{a_{\nu} + b_{\nu} + 2\sqrt{a_{\nu}b_{\nu}}} = \sqrt{a_{\nu}} + \sqrt{b_{\nu}},$$
$$\sqrt{4c_{\nu+2}} = \sqrt{2(a_{\nu+1} - b_{\nu+1})} = \sqrt{a_{\nu} + b_{\nu} - 2\sqrt{a_{\nu}b_{\nu}}} = \sqrt{a_{\nu}} - \sqrt{b_{\nu}},$$

thus arises the formulas

$$\sqrt{a_{\nu}} + \sqrt{b_{\nu}} = 2\sqrt{a_{\nu+2}}, \quad \sqrt{a_{\nu}} - \sqrt{b_{\nu}} = 2\sqrt{c_{\nu+2}} \quad (v >=< 0) \tag{23}$$

and analogously

$$\sqrt{\bar{a}_{\nu}} + \sqrt{\bar{c}_{\nu}} = 2\sqrt{\bar{a}_{\nu+2}}, \quad \sqrt{\bar{a}_{\nu}} - \sqrt{\bar{c}_{\nu}} = 2\sqrt{\bar{b}_{\nu+2}} \tag{24}$$

Both relations (8) and (12) can thus be expanded in the following way:

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\bar{a}_{-n}}{2^n} = \lim_{n\to\infty} \frac{\bar{b}_{-n}}{2^n} = M(a,b) = M(a_n, b_n) \\ = M(a_{-n}, b_{-n}) = 2^{-n}M(\bar{a}_{-n}, \bar{b}_{-n}) = 2^n M(\bar{a}_n, \bar{b}_n);\end{aligned} \tag{25}$$

$$\begin{aligned}\lim_{n\to\infty} \bar{a}_n = \lim_{n\to\infty} \bar{c}_n = \lim_{n\to\infty} \frac{a_{-n}}{2^n} = \lim_{n\to\infty} \frac{c_{-n}}{2^n} = M(a,c) = M(\bar{a}_n, \bar{c}_n) \\ = M(\bar{a}_{-n}, \bar{c}_{-n}) = 2^{-n}M(a_{-n}, c_{-n}) = 2^n M(a_n, c_n)^3;\end{aligned} \tag{26}$$

4. Trigonometric Construction/Delineation/Demonstration/Diagram of the Algorithm [X. 214]

A trigonometric expression of the algorithm of the agM can be given, which is usually most suitable for calculation. We set:

$$\frac{b}{a} = \cos 2M, \quad \frac{b_1}{a_1} = \cos 2M_1, \quad \frac{b_2}{a_2} = \cos 2M_2 \dots \tag{27}$$

then:

$$\frac{a_1}{a} = \tfrac{1}{2}(1 + \cos 2M) = \cos^2 M, \quad \frac{a_2}{a_1} = \cos^2 M_1, \quad \frac{a_3}{a_2} = \cos^2 M_2, \dots$$

Between M, M_1, M_2 exist easily assignable relations, namely

$$\begin{gathered}\tan^2 M = \frac{1 - \frac{a_1}{a}}{\frac{a_1}{a}} = \frac{a-b}{a+b} = \frac{1}{a_1}\sqrt{a_1^2 - b_1^2} = \sin 2M_1, \\ \tan^2 M_1 = \sin 2M_2, \quad \tan^2 M_2 = \sin 2M_3, ...\end{gathered} \tag{28}$$

Thus a_1, b_1, a_2, b_2 are successively calculated with the following Schema:

$$\begin{gathered}\frac{b}{a} = \cos 2M, \quad \frac{a_1}{a} = \cos^2 M, \\ \sin 2M_1 = tan^2 M, \frac{b_1}{a_1} = \cos 2M_1, \quad \frac{a_2}{a_1} = \cos^2 M_1, \\ \sin 2M_2 = tan^2 M_1, \frac{b_2}{a_2} = \cos 2M_2, \quad \frac{a_3}{a_2} = \cos^2 M_2, \dots\end{gathered} \tag{29}$$

Then the agM is directly given through

$$M(a,b) = a_\infty = a\cos^2 M \cos^2 M_1 \cos^2 M_2 \cos^2 M_3 \dots \tag{30}$$

5. The Power Series for M(1,1+x). [III. 365]

We posit now the problem, to find a series built on the [fortschreitende] powers of x for the mean $M(1,1+x)$. Since $M(1,1) = 1$, we apply:

$$M(1,1+x) = 1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + ...,$$

where α_ν are constant coefficients; their determination carried out by the functional equation arising from (8).

$$M(1,1+x) = M(1 + \frac{x}{2}, \sqrt{1+x}). \tag{31}$$

Namely, we set:

$$x = 2t + t^2$$

then, from (31) follows

$$M(1,1+x) = (1+t)M(1+\frac{\frac{1}{2}t^2}{1+t},1),$$

thus

$$\begin{aligned}1 + \alpha_1(2t+t^2) + \alpha_2(2t+t^2)^2 + \alpha_3(2t+t^2)^3 + ...\\ = 1 + t + \alpha_1\frac{1}{2}t^2 + \alpha_2\frac{\frac{1}{4}t^4}{1+t} + \alpha_3\frac{\frac{1}{8}t^6}{(1+t)^2} + ...\end{aligned}$$

from which, by gathering like coefficients, the following equations for $\alpha_1,\ \alpha_2,\ \alpha_3$...arise:

$$\begin{aligned}2\alpha_1 &= 1,\\ 4\alpha_2 + \alpha_1 &= \tfrac{1}{2}\alpha_1,\\ 8\alpha_3 + 3\alpha_2 &= 0,\\ 16\alpha_4 + 12\alpha_3 + \alpha_2 &= \tfrac{1}{4}\alpha_2, ...\end{aligned}$$

and from that is acquired:

$$\alpha_1 = \frac{1}{2},\ \alpha_2 = -\frac{1}{16},\ \alpha_3 = \frac{1}{32},\ \alpha_4 = -\frac{21}{1024},\ \alpha_5 = \frac{31}{2048},\ \alpha_5 = \frac{195}{16384}, ...$$

Thus the sought series is

$$\begin{aligned}M(1,1+x) = 1 + \frac{1}{2}x - \frac{1}{16}x^2 + \frac{1}{32}x^3\\ - \frac{21}{1024}x^4 + \frac{31}{2048}x^5 - \frac{195}{16384}x^6 \pm ... \quad (32)\end{aligned}$$

However, since their coefficients adhere to no clear law, we ignore this series and adopt another method, which warrants a fortunate outcome.

6. The Power Series for $M(1-x,1+x)$ and $\frac{1}{M(1-x,1+x)}$.

[III.366-69.]

We solve the problem, by the power series built on x.

$$M(1+x,1-x) = (1-x)M(1+\frac{2x}{1-x},1),$$

and hence the sought series is obtained from (32), in which $1+\frac{2x}{1-x}$ is substituted for x; by which arises:

$$\begin{aligned}M(1,1+\frac{2x}{1-x}) = 1 + \frac{x}{1-x} - \tfrac{1}{4}\frac{x^2}{(1-x)^2} + \tfrac{1}{4}\frac{x^3}{(1-x)^3} - \tfrac{21}{64}\frac{x^4}{(1-x)^4} \pm ...\\ = 1 + x + \tfrac{3}{4}x^2 + \tfrac{3}{4}x^3 + \tfrac{43}{64}x^4 + \tfrac{43}{64}x^5 + \tfrac{161}{256}x^6 + ...\end{aligned}$$

and from here:

$$M(1+x,1-x) = 1 - \frac{1}{4}x^2 - \frac{5}{64}x^4 - \frac{11}{256}x^6 - \frac{469}{16384}x^8 - ... \tag{33}$$

The coefficients of this series can also be established/acquired/determined independently of (32) in the following way. It is said:

$$x = \frac{2t}{1+t^2},$$

then

$$\begin{aligned} M(1+x,1-x) = M(1,\sqrt{1-x^2}) &= M(1,\frac{1-t^2}{1+t^2}) \\ &= \frac{1}{1+t^2}M(1+t^2,1-t^2). \end{aligned} \tag{34}$$

We add on to this:

$$M(1+x,1-x) = 1 + \beta_1 x^2 + \beta_2 x^4 + \beta_3 x^6 + \beta_4 x^8 + ...,$$

–it is immediately obtained, that due to the invariance of $M(1+x,1-x)$ under a permutation of x with $-x$, the odd powers of x can not occur–, Thus will:

$$\begin{aligned} (1+t^2)\left\{1 + \beta_1\left(\frac{2t}{1+t^2}\right)^2 + \beta_2\left(\frac{2t}{1+t^2}\right)^4 + \beta_3\left(\frac{2t}{1+t^2}\right)^6 + ...\right\} \\ = 1 + \beta_1 t^4 + \beta_2 t^8 + \beta_3 t^{12} + \beta_4 t^{16} + \dots, \end{aligned}$$

from which by the equations of the coefficients follows:

$$\begin{aligned} 1 + 4\beta_1 &= 0 \\ -4\beta_1 + 16\beta_2 &= \beta_1, \\ 4\beta_1 - 48\beta_2 + 64\beta_3 &= 0, \\ -4\beta_1 + 96\beta_2 - 320\beta_3 + 256\beta_4 &= \beta_2, ... \end{aligned}$$

and from here obtain:

$$\beta_1 = -\frac{1}{4},\ \beta_2 = -\frac{5}{64},\ \beta_3 = -\frac{11}{256},\ \beta_4 = -\frac{469}{16384}, ...$$

thus we again come back to the series (33).

In this series as well, the coefficients are subject to no single law; on the other hand, divide it into one and get:

$$\begin{aligned} \frac{1}{M(1+x,1-x)} &= 1 + \frac{1}{4}x^2 + \frac{9}{64}x^4 + \frac{25}{256}x^6 + \frac{1225}{16384}x^8 +, ... \\ &= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^6 + \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2 x^6 + ..., \end{aligned} \tag{35}$$

where the coefficients thus progress, following a very simple law. However this method, by means of which we acquired this quite nice result, depends solely on the power of proof of a conclusion by induction; the deductive proof is the objective of the following consideration.

We make the proposition:

$$\frac{1}{M(1+x,1-x)} = 1 + \alpha_3 x^2 + \alpha_5 x^4 + ... + \alpha_{2k+1} x^{2k} + ...$$

The functional equation (34) then yields:

$$\frac{1}{t+t^2}\ \frac{1}{M(1+x,1-x)} = \frac{1}{M(1+t^2,1-t^2)}$$

thus

$$\begin{aligned}(1+t^2)^{-1} + \alpha_3 4t^2(1+t^2)^{-3} + \alpha_5 16t^4(1+t^2)^{-5} + ...\\ \alpha_{2k+1}4^k t^{2k}(1+t^2)^{-(2k+1)} + ...\\ = 1 + \alpha_3 t^4 + \alpha_5 t^8 + ... + \alpha_{2k+1} t^{4k} + ...\end{aligned}$$

It is however in general:

$$(1+t^2)^{-\lambda} = \sum_{n=0}^{\infty} \binom{-\lambda}{n} t^{2n},$$

$$\begin{aligned}\binom{-\lambda}{n} &= \frac{1}{n!}(-\lambda)(-\lambda-1)...(-\lambda-n+1) = (-1)^n \binom{\lambda+n-1}{n}\\ &= (-1)^n \binom{n+\lambda-1}{\lambda-1};\end{aligned}$$

thus becomes:

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n t^{2n} + 4\alpha_3 t^2 \sum_{n=0}^{\infty}(-1)^n \binom{n+2}{2} t^{2n} + 16\alpha_5 t^4 \sum_{n=0}^{\infty}(-1)^n \binom{n+4}{4} t^{2n} + ...\\ +4^k \alpha_{2k+1} t^{2k} \sum_{n=0}^{\infty}(-1)^n \binom{n+2k}{2k} t^{2n} + ...\\ = 1 + \alpha_3 t^4 + \alpha_5 t^8 + ... + \alpha_{2k+1} t^{4k} + ...;\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n t^{2n} \Bigg\{1 - 4\alpha_3 \binom{n+1}{2} + 16\alpha_5 \binom{n+1}{4} + ...\\ +(-1)^k 4^k \alpha_{2k+1} \binom{n+k}{2k} + ... + (-1)^n 4^n \alpha_{2n+1}\Bigg\}\\ = \sum_{n=0}^{\infty} \alpha_{2n+1} t^{4n} \qquad (\alpha_1 = 1)\end{aligned}$$

From this follows two different classes of equations for the determination of α_ν, namely

$$1 - 4\alpha_3\binom{n+1}{2} + 16\alpha_5\binom{n+2}{4} + ... + (-1)^k 4^k \alpha_{2k+1}\binom{n+k}{2k}$$
$$+ ... + (-1)^n 4^n \alpha_{2n+1} = \begin{cases} 0 & (n \text{ odd}) \\ \alpha_{n+1} & (n \text{ even}) \end{cases}$$

Out of this is obtained the relations:

$$n_2\alpha_{n+1} - (n-1)_2\alpha_{n-1} = \left\{n^2 - (n-1)^2\right\}$$
$$-4\alpha_3\left\{n^2\binom{n+1}{2} - (n-1)^2\binom{n-1}{2}\right\} + ...$$
$$+(-1)^k 4^k \alpha_{2k+1}\left\{n^2\binom{n+k}{2k} - (n-1)^2\binom{n+k-2}{2k}\right\} + ...$$
$$-(-1)^n 4^{n-1}\alpha_{2n-1}n^2\binom{2n-1}{2n-2} + (-1)^n 4^n \alpha_{2n+1}n^2.$$

The parenthetical expression simplifies by the identity:

$$n^2 - (n-1)^2 = 2n - 1,$$

$$n^2\binom{n+k}{2k} - (n-1)^2\binom{n+k-2}{2k}.$$
$$= \binom{n+k-2}{2k-2}\frac{n^2(n+k)(n+k-1) - (n-1)^2(n-k)(n-k-1)}{(2k-1)2k}$$
$$= \binom{n+k-2}{2k-2}(2n-1)\frac{n^2(2n+k) - n(2k+1) + k(k+1)}{(2k-1)2k}$$

thus is found:

$$n^2\alpha_{n+1} - (n-1)^2\alpha_{n-1} = (2n-1)\left\{1 - 4\alpha_3\frac{3n^2 - 3n + 2}{1 \cdot 2}\right.$$
$$+ 16\alpha_5\binom{n}{2}\frac{5n^2 - 5n + 6}{3 \cdot 4} \pm ...$$
$$+ (-1)^k 4^k \alpha_{2k+1}\binom{n+k-2}{2k-2}\frac{n^2(2k+1) - n(2k+1) + k(k+1)}{(2k-1)2k} + ...$$
$$\left. +(-1)^n 4^n \alpha_{2n+1}\frac{n^2}{2n-1}\right\}.$$

however, further applies

$$(2k+1)\binom{n+k-1}{2k} = (2k+1)\binom{n+k-2}{2k-2}\frac{(n-k)(n+k-1)}{(2k-1)2k}$$
$$= \binom{n+k-2}{2k-2}(2k+1)\frac{n^2-n+k(1-k)}{(2k-1)2k},$$
$$= \binom{n+k-2}{2k-2}\frac{n^2(2k+1)-n(2k+1)+k(k+1)-2k^3}{(2k-1)2k},$$

consequently:

$$\binom{n+k-2}{2k-2}\frac{n^2(2k+1)-n(2k+1)+k(k+1)}{(2k-1)2k},$$
$$= (2k+1)\binom{n+k-2}{2k} + \frac{k^2}{2k-1}\binom{n+k-2}{2k-2},$$

and thus by:

$$\begin{aligned}
&n^2\alpha_{n+1} - (n-1)^2\alpha_{n-1} \\
&= (2n-1)\,\{\ 1 - 3\cdot 4\binom{n}{2}\alpha_3 + 5\cdot 16\binom{n+1}{4}\alpha_5 \pm \ldots \\
&\quad - 4\cdot\frac{2}{1\cdot 2}\alpha_3 + \tfrac{4}{3}\cdot 16\binom{n}{2}\alpha_5 \pm \ldots \\
&\quad + (-1)^k 4^k(2k+1)\binom{n+k-1}{2k}\alpha_{2k+1} \\
&\quad - (-1)^k 4^{k+1}(2k+3)\binom{n+k}{2k+2}\alpha_{2k+3} \pm \ldots \\
&\quad + (-1)^k 4^k\frac{k^2}{2k-1}\binom{n+k-2}{2k-2}\alpha_{2k+1} \\
&\quad -(-1)^k 4^{k+1}\frac{(k+1)^2}{2k+1}\binom{n+k-1}{2k}\alpha_{2k+3} \pm \ldots \Big\} \\
&= (2n-1)\left\{(4\alpha_3 - 1) + 4\binom{n}{2}\tfrac{1}{3}(16\alpha_5 - 9\alpha_3) \pm \ldots\right. \\
&\quad - (-1)^k 4^k\binom{n+k-1}{2k}\frac{1}{2k+1}\left((2k+2)^2\alpha_{2k+3} - (2k+1)^2\alpha_{2k+1}\right) \\
&\quad \left.\pm \ldots + (-1)^n 4^{n-1}\frac{1}{2n-1}\left(4n^2\alpha_{2n+1} - (2n-1)^2\alpha_{2n-1}\right)\right\}.
\end{aligned} \tag{36}$$

This equation applies for even and odd n, if the α with even index are set equal to zero. Out of it follows for $n = 1, 2, 3, 4...$ successively, the relations:

$$\begin{aligned} 4\alpha_3 - 1 &= 0, \\ 16\alpha_5 - 9\alpha_3 &= 0, \\ 36\alpha_7 - 25\alpha_5 &= 0, ... \\ (2k)^2\alpha_{2k+1} - (2k-1)^2\alpha_{2k-1} &= 0, ... \end{aligned}$$

from which is obtained:

$$\alpha_3 = \tfrac{1}{4}, \quad \alpha_5 = \frac{1 \cdot 9}{4 \cdot 16}, \quad \alpha_7 = \frac{1 \cdot 9 \cdot 25}{4 \cdot 16 \cdot 36},$$

$$\alpha_{2k+1} = \left(\frac{1 \cdot 3 \cdot 5 ... (2k-1)}{2 \cdot 4 \cdot 6 ... 2k} \right)^2; \tag{37}$$

by this the suspected law of coefficients of the series (35)is in fact deomonstrated. [4]

7. The Differential Equation

[III.370]

The just attained concise series (35) sufices as a simple as well as elegant differential equation, which is of extreme importance [einschneidender Bedeutung] for the entire theory. For the moment, we set:

$$\frac{1}{M(1+x, 1-x)} = y = 1 + \frac{1}{4}x^2 + \frac{1 \cdot 9}{4 \cdot 16}x^4 + ... \\ + \frac{1 \cdot 9 ... (2k-1)^2}{4 \cdot 16 ... (2k)^2}x^{2k} + ...,$$

then:

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$$
\begin{aligned}
x\frac{dy}{dx} =& \frac{1}{2}x^2 + \frac{1}{4}\cdot\frac{9}{4}x^4 + ... \\
&+ \frac{1\cdot 9...(2k-3)^2}{4\cdot 16...(2k-2)^2}\cdot\frac{(2k-1)^2}{2k}x^{2k} + ..., \\
x^2\frac{d^2y}{dx^2} =& \frac{1}{2}x^2 + \frac{1}{4}\cdot\frac{27}{4}x^4 + ... \\
&+ \frac{1\cdot 9...(2k-3)^2}{4\cdot 16...(2k-2)^2}\cdot\frac{(2k-1)^3}{2k}x^{2k} + ..., \\
x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} =& x^2 + \frac{1}{4}\cdot 9x^4 + ... \\
&+ \frac{1\cdot 9...(2k-3)^2}{4\cdot 16...(2k-2)^2}\cdot(2k-1)^2x^{2k} + ..., \\
x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y =& 1 + \frac{9}{4}x^2 + \frac{1\cdot 9}{4\cdot 16}\cdot 25x^4 + ... \\
&+ \frac{1\cdot 9...(2k-3)^2}{4\cdot 16...(2k-2)^2}\cdot\frac{(2k-1)^2}{(2k)^2}(2k+1)x^{2k} + ...,
\end{aligned}
$$

then it is identical:

$$\frac{(2k-1)^3}{2k} + 3\frac{(2k-1)^2}{2k} + \frac{(2k-1)^2}{(2k)^2} = \frac{(2k-1)^2}{(2k)^2}(2k+1)^2.$$

Thus is recognized the existence of the Relation:

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{x^2}\left(x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right),$$

That is, y satisfies the Differential equation

$$(x^3-x)\frac{d^2y}{dx^2} + (3x^2-1)\frac{dy}{dx} + xy = 0.^5 \tag{38}$$

Other than $M(1+x,1-x)^{-1}$ these differential equations have a further particular/individual [particulaeres] integral. They evidently remain unchanged if x^2 is replaced with $1-x^2$. For we set

$$\xi^2 = 1-x^2,$$

then

$$\frac{dy}{dx} = -\frac{\sqrt{1-\xi^2}}{\xi}\frac{dy}{d\xi},\ \frac{d^2y}{dx^2} = \frac{1-\xi^2}{\xi^2}\frac{dy}{d\xi^2} - \frac{1}{\xi^2}\frac{dy}{d\xi},$$

thus

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$$(x^3 - x)\frac{d^2y}{dx^2} + (3x^2 - 1)\frac{dx}{dy} + xy$$
$$= \frac{1}{\xi}\sqrt{1-\xi^2}\left\{\xi(\xi^2-1)\frac{d^2y}{dx^2} + (3\xi^2-1)\frac{dy}{d\xi} + \xi y\right\},$$

that is, (37) turns into

$$(\xi^3 - \xi)\frac{d^2y}{dx^2} + (3\xi^2 - 1)\frac{dy}{d\xi} + \xi y = 0.$$

Now

$$y = M(1+x, 1-x)^{-1} = M(1, \sqrt{1-x^2})^{-1}$$

is a solution from (37), we obtain thus a twice/doubly particular [particulaeres] integral

$$\eta = M(1+\xi, 1-\xi)^{-1} = M(1, \sqrt{1-\xi^2})^{-1} = M(1, x)^{-1}.$$

Das allgemeine...

$$c_1 M(1+x, 1-x)^{-1} + c_2 M(1, x)^{-1}. \tag{39}$$

8. The complete Elliptical Integral of the first class

As is generally known, these relations hold

$$\int_0^\pi \cos^2\phi \; d\phi = \frac{1}{2}\pi,$$
$$\int_0^\pi \cos^4\phi \; d\phi = \frac{1\cdot 3}{2\cdot 4}\pi, ...$$
$$\int_0^\pi \cos^{2n}\phi \; d\phi = \frac{1\cdot 3...(2n-1)}{2\cdot 4...2n}\pi.$$

Thence,

$$\int_0^\pi \frac{d\phi}{\sqrt{1-k^2\cos^2\phi}} \tag{40}$$

$$= \int_0^\pi (1 + \frac{1}{2}k^2\cos^2\phi + \frac{1\cdot 3}{2\cdot 4}k^4\cos^4\phi + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6\cos^6\phi + ...)d\phi$$
$$= \pi\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right) k^6 + ...\right\}$$
$$= \pi M(1+k, 1-k)^{-1} = \pi M(1, \sqrt{1-k^2})^{-1}.$$

We have thus the reciprocal value of the agM used in combination/connection with the complete elliptical integral if the first kind.[6] Naturally this method can also treat integrals of a more general form. For example, if $\gamma > 0$, the integral

$$\frac{1}{\pi}\int_0^\pi \frac{\alpha}{\sqrt{\beta-\gamma\cos^2\phi}}d\phi = \frac{\alpha}{\sqrt{\beta}}\frac{1}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{1-\frac{\gamma}{\beta}\cos^2\phi}}$$
$$= \frac{\alpha}{\sqrt{\beta}}M\left(1, \sqrt{1-\frac{\gamma}{\beta}}\right)^{-1} = M\left(\frac{\sqrt{\beta}}{\alpha}, \frac{\sqrt{\beta-\gamma}}{\alpha}\right)^{-1},$$

thus equals the reciprocal agM of the reciprocal maximum and minimum of the integrands. In this form the formula can also be assigned/carried out in the case that $\gamma < 0$. That is, we set $\psi = \frac{\pi}{2} - \phi$, then

$$\frac{1}{\pi}\int_0^\pi \frac{\alpha}{\sqrt{\beta-\gamma\cos^2\phi}}d\phi = \frac{1}{\pi}\int_0^\pi \frac{\alpha}{\sqrt{(\beta-\gamma)+\gamma\cos^2\psi}}d\psi$$
$$= M\left(\frac{\sqrt{\beta}}{\alpha}, \frac{\sqrt{\beta-\gamma}}{\alpha}\right)^{-1} \qquad (\gamma >=< 0). \quad (41)$$

Also finally

$$\frac{1}{\pi}\int_0^\pi \frac{\alpha}{\sqrt{\beta+\gamma\cos\phi}}d\phi = 2\int_0^{\frac{\pi}{2}} \frac{\alpha}{\sqrt{\beta-\gamma+2\gamma\cos^2\psi}}d\psi \quad (\psi = \frac{1}{2}\phi)$$
$$= \int_0^\pi \frac{\alpha}{\sqrt{\beta-\gamma+2\gamma\cos^2\psi}}d\psi = \pi M\left(\frac{\sqrt{\beta-\gamma}}{\alpha}, \frac{\sqrt{\beta+\gamma}}{\alpha}\right)^{-1}. \quad (42)$$

It is recognized from this, that the value of a complete elliptical integral has to calculated by the following rule: One forms the maximum and minimum value v and v' of the integrand in the interval of integration, then the value of the integral is equal to

$$\pi M\left(\frac{1}{v}, \frac{1}{v'}\right)^{-1}.$$

From this it becomes clear, that if various expressions of the form $\frac{\alpha}{\sqrt{\beta-\gamma\cos\phi}}$ in the integral area [Integrationsgebiet] have the same extrema, their complete integrals come to the same value.

The solved[geloeste] problem can be generalized, in which to ask according to the value of the *undermined* elliptical integral

$$\frac{1}{\pi}\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\cos^2\phi}}$$

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It would be advantageous to proceed such that that the integrand with the cosines of multiple is developed:

$$\frac{d\phi}{\sqrt{1-k^2+\cos^2\phi}} = P_0 + 2P_1\cos 2\phi + 2P_2\cos 4\phi + 2P_3\cos 6\phi + ...,$$

where the P_ν depend only on k. Then

$$\int_0^\phi \frac{d\phi}{\sqrt{1-k^2+\cos^2\phi}} = P_0\phi + P_1\sin 2\phi + \frac{1}{2}P_2\sin 4\phi + \frac{1}{3}P_3\sin 6\phi + ..., \quad (43)$$

and thus still attained only from the determination of the P_ν. The first coefficient we already know:

$$P_0 = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{1-k^2\cos^2\phi}} = M(1,\sqrt{1-k^2})^{-1}; \quad (44)$$

for the determination of other coefficients, we will later give a simple method and similarly a generalized proof of this phrase [Saetze], which is acquired from many natural[urspruenglicheren] principles.[7]

There are always people, that do not know of the sublimity of the eternal truths and their divine beauty and therefore have only learned to appreciate the value of mathematical investigations from their applicability in the field of applied science; the above devotement will have the use, to make these people more pleased at our investigation. In fact, it is generally known, of how great a utility such a rapidly convergent progression, as that developed from the above sentences, is in physical astronomy or the theory of planetary perturbations

11. The connection with the Ellipse quadrant

An ellipse is presented with semi axes a, b ($a > b$); the length of a ellipse quantrant is to be q. We then denote

$$k = \sqrt{\frac{a^2-b^2}{a^2}} = \frac{c}{a}; \quad k' = \frac{b}{a}, \quad k^2 + k'^2 = 1, \quad (45)$$

then, as is generally know, q is educible through a complete ellipse integral of two classes:

$$\begin{aligned} q &= a\int_0^1 \frac{(1-k^2x^2)}{(1-x^2)(1-k^2x^2)}dx \\ &= a\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left\{1 - \frac{1}{2}k^2x^2 - \frac{1}{8}k^4x^4 - \frac{1}{16}k^6x^6 - \frac{5}{128}k^8x^8 - ...\right\}, \end{aligned} \quad (46)$$

and in general

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$$\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}dx = \frac{1\cdot 3\cdot 5...(2n-1)}{2\cdot 4\cdot 6...\ 2n}\cdot\frac{\pi}{2}$$

we find for q the series expansion [8]

$$q = a\frac{\pi}{2}\left\{1-\left(\frac{1}{2}\right)^2\frac{k^2}{1}-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3}\right. \\ \left. -\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^6}{5}-...-\left(\frac{1\cdot 3...(2n-1)}{2\cdot 4...\ 2n}\right)^2\frac{k^{2n}}{2n-1}-...\right\}. \quad [X.178] \tag{47}$$

Similarly as the integral of the first class can also express this *Integral of two classes* in a simple manner through the agM. We generate namely following (35)

$$M(1,k')^{-1} = M(1,\sqrt{1-k^2})^{-1} = 1+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+...+\left(\frac{1\cdot 3...(2n-1)}{2\cdot 4...\ 2n}\right)^2 k^{2n}+...,$$

as is

$$k\frac{d}{dk}M(1,k')^{-1}+M(1,k')^{-1} = 1+\sum_{n=1}^{\infty}\left(\frac{1\cdot 3...(2n-1)}{2\cdot 4...\ 2n}\right)^2(2n+1)k^{2n},$$

$$(1-k^2)\left\{k\frac{d}{dk}M(1,k')^{-1}+M(1,k')^{-1}\right\} = 1-\sum_{n=1}^{\infty}\left(\frac{1\cdot 3...(2n-1)}{2\cdot 4...\ 2n}\right)^2\frac{k^{2n}}{(2n-1)},$$

with which follows from (69) the relation

$$q = a\frac{\pi}{2}(1-k^2)\left\{k\frac{d}{dk}M(1,k')^{-1}+M(1,k')^{-1}\right\}, \quad [X.178] \tag{48}$$

i.e. the sought equation. We can give it a convenient form for calculation. From (67) is

$$dk = -\frac{k'}{k}dk' \tag{49}$$

thus is obtained

$$\frac{d}{dk}M(1,k')^{-1} = \frac{k}{k'}M(1,k')^{-2}\frac{dM(1,k')}{dk'},$$
$$q = a\frac{\pi}{2}M(1,k')^{-1}\left\{k'^2+k^2k'^2M(1,k')^{-1}\frac{dM(1,k')}{dk'}\right\},$$

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and then from (66) is

$$\frac{dM(1,k')}{dk'} = \frac{M(1,k')}{2k'}\left\{1 - \frac{1}{2}\frac{a-a_1}{a_1} - \frac{1}{4}\frac{a-a_1}{a_1}\frac{a_1-a_2}{a_2}...\right\}$$

Finally, we acquire

$$\begin{aligned} q = a\frac{\pi}{2}M(1,k')^{-1}\times \\ \times\left\{1 - \frac{k^2}{2}(1 + \frac{1}{2}\frac{a-a_1}{a_1} + \frac{1}{4}\frac{a-a_1}{a_1}\frac{a_1-a_2}{a_2} + ...)\right\}. \quad [X.178] \end{aligned} \tag{50}$$

The formula (70) is the so called class relation[*Klassenbeziehung*] between the complete elliptical integral of first and second kind.[9]

31. Construction of the continuous fraction
[X. 222 .223]

The entire investigation of the preceding paragraphs was based upon the analysis of case I, which we had accomplished through analysis of the substitution S in the product of $S_{m,n}$. There is however still one other operation, to complete this case, in which a continuous fraction is developed. For that, we write:

$$t' = \frac{\alpha t - \beta i}{\gamma i t + \delta} = \frac{1}{\frac{\gamma i t + \delta}{\alpha t - \beta i}}$$

If α_1 is a still to be determined whole number, then we set:

$$\gamma = 2a_1\alpha + \gamma_1;\ \delta_1 = \delta - 2a_1\beta,$$

then

$$t' = \frac{1}{2a_1 i + \frac{-\gamma_1 t + \delta_1 i}{\alpha i t + \beta}}$$

It is seen immediately, that the [following] congruences apply:

$$\gamma_1 \equiv \gamma \equiv 0 \pmod 2,\ \delta_1 \equiv \delta \equiv 1 \pmod 2,$$

and that the determinate of the last fraction is:

$$-\beta\gamma_1 + \alpha\delta_1 = \alpha\delta - \beta\gamma = 1$$

We determine the number a_1 such that

$$|\delta_1| < |\beta|$$

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We now set analogously:

$$\alpha = -2a_2\gamma_1 + \alpha_1; \quad \beta_1 = \beta + 2a_2\delta_1,$$

then

$$\frac{-\gamma_1 t + \delta_1 i}{\alpha i t + \beta} = \frac{1}{\dfrac{\alpha i t + \beta}{-\gamma_1 t + \delta_1 i}} = \frac{1}{2a_2 i + \dfrac{\alpha_1 t - \beta_1 i}{\gamma_1 i t + \delta_1}},$$

where

$$\alpha_1 \equiv \alpha \equiv 1 \pmod 4, \;\; \beta \equiv \beta \equiv 0 \pmod 2,$$
$$\alpha_1\delta_1 - \beta_1\gamma_1 = \alpha\delta_1 - \beta\gamma_1 - 1$$

we choose the whole number so that

$$|\beta_1| < |\delta_1| < |\beta| \tag{51}$$

It is then thus combined/consolidated

$$t' = \cfrac{1}{2a_1 i + \cfrac{1}{2a_2 i + \cfrac{\alpha_1 t - \beta_1 i}{\gamma_1 i t + \delta_1}}}$$

where the last fraction itself satisfies convergence and determinant conditions as that from which we has proceeded. Only with this fraction can we likewise proceed and obtain with apropriate choice the whole numbers $a_3, \; a_4$

$$\frac{\alpha_1 t - \beta_1 i}{\gamma_1 i t + \delta_1} = \cfrac{1}{2a_3 i + \cfrac{1}{2a_4 i + \cfrac{\alpha_2 t - \beta_2 i}{\gamma_2 i t + \delta_2}}},$$
$$\alpha_2 \equiv \quad \pmod 4, \;\; \beta_2 \equiv \gamma_2 \equiv 0 \pmod 2, \;\; \delta \equiv 1 \pmod 2$$
$$\alpha_2\delta_2 - \beta_2\gamma_2 = 1,$$
$$|\beta_2| < |\delta_2| < |\beta_1|\,.$$

Through continuation of the procedure, eventually a fraction for the $\beta_n u = 0$, thus $\alpha_n u = \delta_n u = 1$, is obtained, which thus has the form:

$$\frac{t}{\gamma_n u i t + 1} = \frac{1}{2a_{2\nu+1} i + \dfrac{1}{t}}, \quad (a_{2\nu+1} = \frac{1}{2}\gamma_\nu).$$

Therefore each substitution can be represented in the form of a continuous fraction

$$t' = \frac{\alpha t - \beta i}{\gamma i t + \delta} = \cfrac{1}{2a_1 i + \cfrac{1}{2a_2 i + \cfrac{1}{2a_3 i + \cfrac{1}{\ddots + \cfrac{1}{2a_{2\nu+1} i + \cfrac{1}{t}}}}}}, \tag{52}$$

Where the a_k form an odd number/quantity of real whole numbers. Also, inversely, each continuous fraction of the form (75) represents a transformation of the 1st class. Then next is

$$t_0 = \cfrac{1}{2a_{2\nu+1} i + \cfrac{1}{t}} = \frac{t}{2a_{2\nu+1} t + 1}$$

a transformation of the 1st class of determinant 1; then

$$\cfrac{1}{2a_{2\nu-1} i + \cfrac{1}{2a_{2\nu} i + t_0}} = \frac{t_0 + 2a_{2\nu} i}{2a_{2\nu-1} i t_0 + (1 - 4a_{2\nu} a_{2\nu-1})}$$

a further substitution with the same properties, and hence from the iteration of this conclusion arises the fact, that each continuous fraction of the form (75) represents a substitution of the 1st class with the determinant 1.

By means of the representation (75) it is now easy to examine the behavior of $p(t)$, $q(t)$, $r(t)$ with such a the substitution. For that, we set:

$$2a_{2\nu+1} i + \frac{1}{t} = t_{2\nu+1},\ 2a_{2\nu} i + \frac{1}{t_{2\nu+1}} = t_{2\nu}; ...2a_1 i + \frac{1}{t_2} = t_1$$
$$t' = \frac{1}{t_1}.$$

, then following (46), (48):

$$\begin{aligned} p(t') &= \sqrt{t_1} p(t_1) = \sqrt{t_1 t_2} p(t_2) = ... \\ &= \sqrt{t_1 t_2 ... t_{2\nu+1} t}\, p(t), \\ q(t') &= \sqrt{t_1} r(t_1) = i^{-a_1} \sqrt{t_1 t_2} q(t_2) = ... \\ &= i^{a_1 + a_3 + ... + a_{2\nu+1}} \sqrt{t_1 t_2 ... t_{2\nu+1} t}\, q(t), \\ r(t') &= \sqrt{t_1} q(t_1) = \sqrt{t_1 t_2} r(t_2) = ... \\ &= i^{a_2 + a_4 + ... + a_{2\nu}} \sqrt{t_1 t_2 ... t_{2\nu+1} t}\, r(t), \end{aligned} \tag{53}$$

which completely corresponds with our formula (61). One can now always further discuss how to cover the above in connection to the formula (76), however we will not go into this more, since it is the basically the same train of thought.

32. The Quadratic Form. The Fundamental Domain

[III. 386. 477. VIII. 100-101. X.224-226]

There remains an important relation between the theory of linear fractional substitutions treated here and that of the quadratic forms with negative determinants. If F and F' are two quadratic forms with the variables u, v and u', v and whole number coefficients:

$$F = au^2 + 2buv + cv^2,\ F' = a'u'^2 + 2b'u'v' + c'v'^2.$$

then the two forms transform into each other by a whole number coefficient of the variables with determinant 1, thus the two formulas are called *intrinsically equivalent.*[10]

$$u' = \alpha u + \beta v,\ v' = \gamma u + \delta v,\ \alpha\delta - \beta\gamma = 1$$

Evidently, a *necessary* condition for the equivalence of two forms is the equality of their determinants:

$$d = b^2 - ac = d' = b' - a'c'.$$

A quadratic form F is called *definite*, if it always has the same sign (except for $u = v = 0$) for all possible real values of u, v, that is if their roots are not real then its determinant is negative. If now F and F' are two positive definite equivalent forms, that is

$$a > 0,\ a' > 0,\ d = d < 0,$$

from which also follows $c > 0$, $c' > 0$; then between the values $\frac{u}{v}$ and $\frac{u'}{v'}$ for vanishing values of F and F', according to the definition of equivalence a relation of the form:

$$\frac{u'}{v'} = \frac{\alpha\frac{u}{v} + \beta}{\gamma\frac{u}{v} + \delta}$$

with whole reals α, β, γ, δ, the determinant must remain 1. We will hence denote two definite quadratic forms of *one* variable:

$$ax^2 + 2bx + c \text{ and } a'x^2 + 2b'x + c' \tag{54}$$

as equivalent, if the roots, the one $x = ti$ and $x = t'i$ the other, are related by a real linear fractional whole number substitution with the determinant $+1$, that is, if

$$t'i = \frac{\alpha ti + \beta}{\gamma ti + \delta'}$$

or

$$t' = \frac{\alpha t - \beta i}{\gamma ti + \delta},$$

thus t' and t relate by a substitution of the form (54).

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Now if t is any rational value

$$t = \frac{\lambda + \mu i}{-\sigma i + \tau},$$

where $\lambda,\ \mu,\ \sigma,\ \tau$ are whole real numbers and numerators and denominators without common factors. Then a quandratic form can be found, whose root is ti; that is, set:

$$\sigma^2 + \tau^2 = A, \quad \lambda\sigma + \mu\tau = B, \quad \lambda^2 + \mu^2 = C,$$
$$\sqrt{AC - B^2} - \lambda\tau - \mu\sigma = D,$$

then:

$$ti = \frac{(\sigma i + \tau)(\lambda i - \mu)}{\sigma^2 + \tau^2} = \frac{-B + Di}{A} \tag{55}$$

is a root of the positive definite form

$$Ax^2 + 2Bx + C = (A, B, C), \tag{56}$$

which has the negative determinate $B^2 - AC = -D^2$. Now, for this form (81) an equivalent form (a, b, c) can always be found.

$$2\,|b| \leq a \leq c \tag{57}$$

This form is denoted as the *reduced* of the original[11] or also as the simplest equivalent form to (81). To arrive at this reduced form, we proceed with the following Algorithm.

We set

$$D^2 + B^2 = AA_1, B + B_1 = hA_1, \text{ h a whole number}, \tag{58}$$

then we chose $A_1 = C$ and B_1 as the absolute least remainder of $B \pmod{A_1}$, so that thus

$$|B_1| \leq \tfrac{1}{2} A_1$$

Evidently now

$$D^2 + B_1^2 \equiv D^2 + B^2 \equiv 0 \pmod{A_1};$$

we thus set

$$D^2 + B_1^2 = A_1 A_2,$$

and determine B_2 from the equation,

$$B_1 + B_2 = h_1 A_2 \ h_1 \text{ a whole number},$$

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again as absolute least residue of $B_1 \pmod{A_2}$, if not already $A_2 \geq A_1$, so that

$$|B_2| \leq \tfrac{1}{2}A_2$$

Then again

$$D^2 + B_2^2 = 0 \pmod{A_2},$$

and it becomes

$$D^2 + B_2^2 = A_2 A_3,$$

if not already $A_3 \geq A_2$, then we again determine B_3 from

$$B_2 + B_3 = h_2 A_3 h_2 \text{ a whole number},$$

as absolute least residue of $B_2 \pmod{A_2}$, thus

$$|B_3| \leq \tfrac{1}{2}A_3,$$

and continue so forth, from

$$D^2 + B_{\nu-1}^2 = A_{\nu-1}A\nu$$

arises $A_\nu \geq A_{\nu-1}$, what must be carried out according to a finite number of measures [Schritten], since otherwise a not unbroken progression of positive whole numbers $A_1,\ A_2,\ A_3,...$ is obtained where at least all around 1 were smaller than the preceding. The quadratic form

$$(A_k, B_k, A_{k+1}) \quad (k = 1, 2, ...\nu - 1)$$

with the roots

$$t_k i = \frac{-B_k + Di}{A_k}$$

are then all equivelent with $(A, B, A_1) = (A, B, C)$; because

$$t_1 = \frac{D - iB + ihA_1}{A_1} = hi + \frac{A}{D + Bi} = hi + \frac{1}{t},$$

$$t_2 = h_1 i + \frac{1}{t_1},$$

$$\cdots\cdots$$

$$t_{\nu...1} = h_{\nu-2} i + \frac{1}{t_{\nu-2}}, \tag{59}$$

that is $t_k (k = 1, 2, ...\nu - 1)$ relate to each other and with t by linear-broken transformations of the form (54). Thus, in this way, we have produced in a form equivalent to (A, B, C)

$$(A_{\nu-1}, B_{\nu-1}, A_\nu)$$

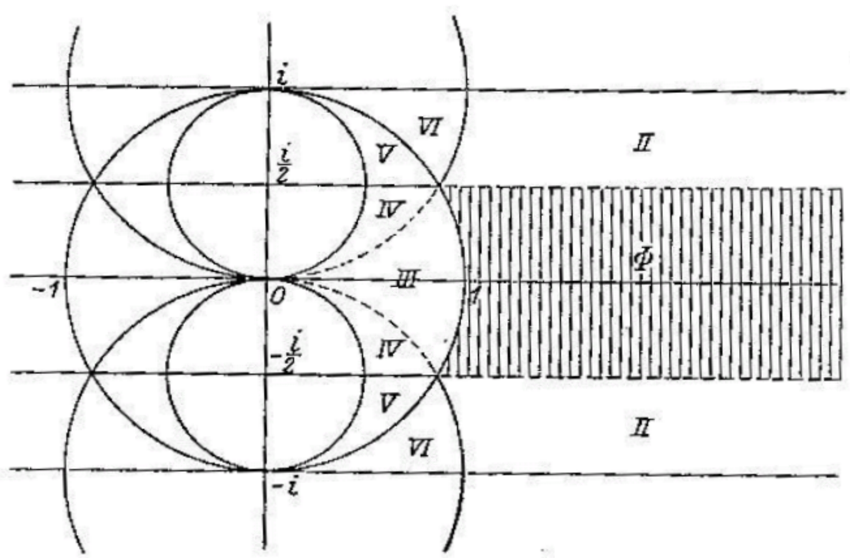
for

$$2\,|B_{\nu-1}| \leq A_{\nu-1} \leq A_\nu$$

which thus presents the reduced from of (81) according to (82). The root t' of this form has therefore the characteristic/feature, that the imaginary part of t' as well as t'^{-1} is less than or equal to $\frac{1}{2}$:

$$\Im(t') \leq \frac{1}{2}, \quad \Im(\frac{1}{t'}) \leq \frac{1}{2}.$$

The domain of the t-plane, in which these points lie, evidently arise out of the part of the the band [Streifens], for which $\Im(t) \leq \frac{1}{2}$, which lies outside of the unit circle, just as the reflection of this part about the unit circle, that is, the dashed bordered region of the t-plane in figure below[12], of which, for our convenience, only the right half of the plane in question appears. Every rational value of t with a positive real part can consequently transform by a substitution of the form (54) into a t-value, which lies in the just identified domain.



The foregoing rationality of t is insignificant, since t is not rational, for instance $t = g + ih$, then the reduction algorithm can likewise be built, in which one perhaps starts from the not whole number form

$$\left(\frac{1}{g}, \frac{h}{g}, \frac{g^2+h^2}{g}\right)$$

with the derminant -1. The domain found can still be somewhat narrowed; that is, since the part that lies in the inside the unit circle is itself situated by reflection of the outer half about the unit circle, thus emerges from the transformation

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$t' = \frac{1}{t}$, and this is also a transformation of the form (54), therefore every value of t presented can be reduced to a t-value which lies inside the hatched [shraffieren] domains Phi, by a substitution of the form (54). From this this domain Phi is designated as the *fundamental domain of the transformation* (54).

From the table (67) is seen, that the quotient of two averaging [summatorischer] functions, with all considered substitutions, can assume only an finite number of different values. Therefore, it is enough to know the values of these functions of their respective quotients in a such domain of the complex t-plane, which can be obtained from all possible t-values with positive real parts by substitution of the form (54). we have just ascertained such a domain in Phi.

We will now investigate the fundamental domain of the substitutions which belong to the first class. Since the entirety of these substitutions form only a part of the entirety of substitutions of (54), their fundamental domain must be greater than Phi and must include Phi in it. In the above given analysis of the I to VI$^t h$ classes of substitutions (see section 29) we have shown, that every substitution of the form (54) can be established from a substitution of the first class, if it is composed with one of the following six substitutions.

$$\begin{gathered}\sigma_1 : t' = t,\ \sigma_2 : t' = t + i, \sigma_3 : t' = \frac{1}{t}, \\ \sigma_4 : t' = \frac{1}{t+i}, \sigma_5 : t' = \frac{t}{it+1}, \sigma_5 : t' = \frac{it+1}{t}.\end{gathered} \tag{60}$$

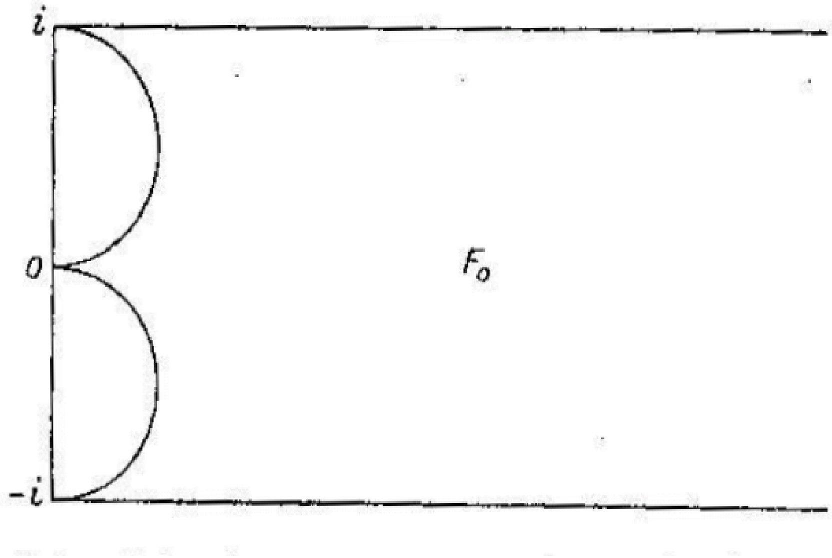
These σ_i are themselves the simplest representation of the six classes. Evidently the fundamental domain F_0 of the substitution which belongs to the 1st class, must comprise all of that domain, which results from the fundamental domain $\Phi = I$ of all substitutions (54) by application of the transformations $\sigma_1 \ldots \sigma_6$; which are however the domains I…VI of the figure (from which we have made use of the sign, that the domain congruent to $2i$ may be shifted). Since then, and only then, if F_0 comprises all these domains, can the entire right t-half plane be generated by substitutions of the 1st class.

F_0 is the fundamental domain of the substitutions of the first class. This fundamental domain will thus be bounded by the lines $t = \pm i$ and the two half circles bounded by $0,\ i$ and $0, -i$; according to this, it exists out of the points t, is for the same time

$$\Im(t) \leq 1, \quad \Im(\frac{1}{t}) \leq 1$$

and both circular bounding bands[Begrenzungsstuecke] arise from strait lines by the reflection about the unit circle, that is, by the transformation $t \to \frac{1}{t}$. Evidently, again, only the right half of t-half comes into question, since only in it is the condition $|x| < 1$ satisfied.

We investigate only the value, which k' as a function of t assumes in the fundamental domain. We maintain, the fundamental domain will map uniquely from the right half of the k-plane, that is, we let t traverse all values of the fundamental domain F_0, thus take k' to be all values with a positive real part



and each only once. We prove this geometrically, in which we study the characteristics of k from the boundaries of the domains. For the strait line bounding band [Begrenzungstuecke] is first of all: $t = \pm i + \tau$, where τ traverses positive real values from 0 to ∞. We represent the value of k at these lines by $\bar{k}'$, then:

$$k' = \left[\frac{q(t)}{p(t)}\right]^2 = \left[\frac{p(t)}{q(t)}\right]^2 = \left[\frac{1 + 2e^{\pi\tau} + 2e^{4\pi\tau} + ...}{1 - 2e^{\pi\tau} + 2e^{4\pi\tau} - ...}\right]^2$$

is real, and if τ' traverses its range, k' traverses all positive real values from 1 to ∞ once. Therefore the two strait $t = \pm i + \tau$ corresponds to the real axis of the k'-plane from 1 to ∞ twice. Both semicircular bounding lines of the fundamental domain emerge now from the just applied, by the transformation $t \to \frac{1}{t}$, thus from them:

$$k' = \left[\frac{q\left(\frac{1}{t}\right)}{p\left(\frac{1}{t}\right)}\right]^2 = \left[\frac{r(t)}{p(t)}\right]^2 = \sqrt{\frac{p(t)^4 - q(t)^4}{p(t)^4}} = \sqrt{1 - \bar{k}'^2};$$

thus the whole imaginary axis from ∞i to $+\infty i$ corresponds to the two semicircles in the k-plane. Therefore, the right k-half plane corresponds to the fundamental domain, in which the straight line from $+1$ to $+\infty$ is doubly counted. The positive real t-axis, which is

$$k' = \left[\frac{1 - 2e^{-\pi t} + 2e^{-4\pi t} \mp ...}{1 + 2e^{-\pi t} + 2e^{-4\pi t} + ...}\right] \leq 1$$

corresponds the band from 0 to 1 of the real axis, so that the halves of the fundamental domain will be mapped directly from a quadrant of the k-plane.

Therefore take each value k'^2 in the fundamental domain once, and with exception of the boundaries, *also taken only once*, that is, the equation

$$k'^2 = \left[\frac{q(t)}{p(t)}\right]^4 = A^2 = const. \tag{61}$$

has one and only one solution in the fundamental domain. [III. 478] [13] k'^2, assumed as a function of t, is called the *Modular Function* and according to this, the theorum, that *the fundamental domain F_0 of the t plane is mapped onto the entire plane of the modular function*, is obtained.

Hence F_0 is designated as the *fundamental domain of the modular function.*

33. Elliptical integral infinite number of values of the agM

34. Lemniscate function

39. transformation of the elliptical integral

40 du/r of ellipse

41. J = integral of ellipse

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