GENERAL EXAMINATION
OF THE
INFINITE SERIES
$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+$ etc.

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(translated from Latin into English by D. Kikuchi, M. A.)

FIRST PART.

INTRODUCTION.

1.

The series which we propose to discuss in this paper can be regarded as a function of four quantities α, β, γ, x, which we shall call its *elements*, distinguishing according to their order α as the first, β as the second, γ as the third, and x as the fourth element. Evidently the first element may be interchanged with the second: so that if we denote our series for shortness' sake by the symbol $F(\alpha, \beta, \gamma, x)$, we have

$$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$$

2.

Assigning definite values to the elements α, β, γ, our series becomes a function of the single variable x, which evidently stops suddenly after the $(1-\alpha)^{\text{th}}$ or $(1-\beta)^{\text{th}}$ term if $\alpha - 1$ or $\beta - 1$ be a negative integer, but in all other cases runs on to infinity. In the former case, the series presents a rational algebraical function, but in the latter case mostly a transcendental function. The third element γ must neither be a negative integer nor $= 0$, in order that we may not have terms infinitely great.

3.

The co-efficients of x^m, x^{m+1} in our series are as

$$1 + \frac{\gamma+1}{m} + \frac{\gamma}{mm} : 1 + \frac{\alpha+\beta}{m} + \frac{\alpha\beta}{mm},$$

so that the greater we take m the nearer they approach the ratio of equality. If therefore we assign to the fourth element x also a definite value, the divergence or convergence depends upon the nature of this value. This is to say, if we assign to x a real value, positive or negative, less than unity, the series will assuredly be convergent if not at once from the beginning, certainly after a certain interval, and will lead to a finite sum which is perfectly determinate. The same thing will happen for imaginary values of x of the form $a + b\sqrt{-1}$ provided $aa + bb < 1$. But on the contrary for a real value of x greater than unity or for an imaginary of the form $a + b\sqrt{-1}$ where $aa + bb > 1$, the series will, if not at once, certainly after a certain interval be necessarily divergent so that we must not speak of its *sum.* Finally for the value $x = 1$ (or more generally for the value of the form $a + b\sqrt{-1}$, where $aa + bb = 1$), the convergence or divergence of the series will depend upon the nature of α, β, γ themselves, of which as well as of the sum of the series for $x = 1$, we shall speak in the Third Section.

It is evident therefore that so long as our function is defined as the sum of a series, our discussion is by its own nature restricted to those cases where the series is truly convergent, and consequently that any enquiry as to the value of the series for a value of x greater than unity is impertinent. But below, from the Fourth Section onward, we shall construct our function on a higher basis and thus render it capable of the most general application.

4.

The differentiation of our series, considering only the element x as variable, leads to a similar function since we have evidently

$$\frac{dF(\alpha, \beta, \gamma, x)}{dx} = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x)$$

The same is true of successive differentiations.

5.

It will be worth while to place here certain functions whic can be reduced to our series and which are of very frequent occurrence in all analysis.

I. $(t+u)^n = t^n F(-n, \beta, \beta, -\frac{u}{t})$

where the element β is arbitrary.

II. $(t+u)^n + (t-u)^n = 2t^n F(-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, \frac{uu}{tt})$

III. $(t+u)^n + t^n = 2t^n F(-n, \omega, 2\omega, -\frac{u}{t})$

ω denoting an infinitely small quantity.

IV. $(t+u)^n - (t-u)^n = 2nt^{n-1}uF(-\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+1, \frac{3}{2}, \frac{uu}{tt})$

V. $(t+u)^n - t^n = nt^{n-1}uF(1-n, 1, 2, -\frac{u}{t})$

VI. $\log(1+t) = tF(1, 1, 2, -t)$

VII. $\log\frac{1+t}{1-t} = 2tF(\frac{1}{2}, 1, \frac{3}{2}, tt)$

VIII. $e^t = F(1, k, 1, \frac{t}{k}) = 1 + tF(1, k, 2, \frac{t}{k}) = 1 + t + \frac{1}{2}ttF(1, k, 3, \frac{t}{k})$ etc.

e denoting the base of the hyperbolic logarithm, k an infinitely large number.

IX. $e^t + e^{-t} = 2F(k, k', \frac{1}{2}, \frac{tt}{4kk'})$

k, k' denoting infinitely large numbers.

X. $e^t - e^{-t} = 2tF(k, k', \frac{3}{2}, \frac{tt}{4kk'})$

XI. $\sin t = tF(k, k', \frac{3}{2}, -\frac{tt}{4kk'})$

XII. $\cos t = F(k, k', \frac{1}{2}, -\frac{tt}{4kk'})$

XIII. $t = \sin t.F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \sin t^2)$

XIV. $t = \sin t.\cos t.F(1, 1, \frac{3}{2}, \sin t^2)$

XV. $t = \tan t.F(\frac{1}{2}, 1, \frac{3}{2}, -\tan t^2)$

XVI. $\sin nt = n\sin t.F(\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+\frac{1}{2}, \frac{3}{2}, \sin t^2)$

XVII. $\sin nt = n\sin t.\cos t.F(\frac{1}{2}n+1, -\frac{1}{2}n+1, \frac{3}{2}, \sin t^2)$

XVIII. $\sin nt = n\sin t.\cos t^{n-1}.F(-\frac{1}{2}n+1, -\frac{1}{2}n+\frac{1}{2}, \frac{3}{2}, -\tan t^2)$

XIX. $\sin nt = n\sin t.\cos t^{-n-1}.F(\frac{1}{2}n+1, \frac{1}{2}n+\frac{1}{2}, \frac{3}{2}, -\tan t^2)$

XX. $\cos nt = F(\frac{1}{2}n, -\frac{1}{2}n, \frac{1}{2}, \sin t^2)$

XXI. $\cos nt = \cos t.F(\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}, \sin t^2)$

XXII. $\cos nt = \cos t^n.F(-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}, -\tan t^2)$

XXIII. $\cos nt = \cos t^{-n}.F(\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}n, \frac{1}{2}, -\tan t^2)$

6.

The preceding functions are algebraical or transcendental depending on logarithms and circle. But by no means on account of *these* , do we undertake our *general* examination but rather for the sake of advancing the theory of higher transcendental functions of which our series comprises a very wide genus. To this, among an infinite number of others, belong the coefficients arising from the development of the function $(aa+bb-2ab\cos\phi)^{-n}$ in series proceeding according to the cosines of the angles $\phi, 2\phi, 3\phi$, &c., of which *in particular* we shall treat more at length on another occasion. These coefficients can be reduced to the form of our series in many ways. Thus, putting

$$(aa+bb-2ab\cos\phi)^{-n} = \Omega = A + 2A'\cos\phi + 2A''\cos 2\phi + 2A'''\cos 3\phi + \text{etc.},$$

we shall have, *firstly* ,

$$A = a^{-2n}F(n, n, 1, \tfrac{bb}{aa})$$

$$A' = na^{-2n-1}bF(n, n+1, 2, \tfrac{bb}{aa})$$

$$A'' = \tfrac{n(n+1)}{1.2}a^{-2n-2}bbF(n, n+2, 3, \tfrac{bb}{aa})$$

$$A''' = \tfrac{n(n+1)(n+2)}{1.2.3}a^{-2n-3}b^3F(n, n+3, 4, \tfrac{bb}{aa})$$

etc.

For if $aa+bb-2ab\cos\phi$ be regarded as a product of $a-br$ into $a-br^{-1}$ (r standing for the quantity $\cos\phi+\sin\phi\sqrt{-1}$), Ω becomes equal to the product of a^{-2n}

$$\text{into } 1 + n\tfrac{br}{a} + \tfrac{n(n+1)}{1.2}\cdot\tfrac{bbrr}{aa} + \tfrac{n(n+1)(n+2)}{1.2.3}\cdot\tfrac{n(n+1)(n+2)b^3r^3}{a^3} + \text{etc.}$$

$$\text{into } 1 + n\tfrac{br^{-1}}{a} + \tfrac{n(n+1)}{1.2}\cdot\tfrac{bbr^{-2}}{aa} + \tfrac{n(n+1)(n+2)}{1.2.3}\cdot\tfrac{n(n+1)(n+2)b^3r^{-3}}{a^3} + \text{etc.}$$

As this product must be identical with

$$A + A'(r+r^{-1}) + A''(rr+r^{-2}) + A'''(r^3+r^{-3}) + \text{etc.}$$ [1]

the values given above are obtained at once.

Secondly, we have

$$A = (aa+bb)^{-n}F(\tfrac{1}{2}n, \tfrac{1}{2}n+\tfrac{1}{2}, 1, \tfrac{4aabb}{(aa+bb)^2})$$

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[1] + etc. omitted in the original. D. K.

$$A' = n(aa+bb)^{-n-1}abF(\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}n+1, 2, \tfrac{4aabb}{(aa+bb)^2})$$

$$A'' = \tfrac{n(n+1)}{1.2}(aa+bb)^{-n-2}aabbF(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{3}{2}, 3, \tfrac{4aabb}{(aa+bb)^2})$$

$$A''' = \tfrac{n(n+1)(n+2)}{1.2.3}(aa+bb)^{-n-3}a^3b^3F(\tfrac{1}{2}n+\tfrac{3}{2}, \tfrac{1}{2}n+2, 4, \tfrac{4aabb}{(aa+bb)^2})$$

etc.

which values are easily deduced from

$$\Omega(aa+bb)^n = 1 + n(r+r^{-1})\tfrac{ab}{aa+bb} + \tfrac{n(n+1)}{1.2}(r+r^{-1})^2\tfrac{aabb}{(aa+bb)^2} + \text{etc.}$$

Thirdly,

$$A = (a+b)^{-2n}F(n, \tfrac{1}{2}, 1, \tfrac{4ab}{(a+b)^2})$$

$$A' = n(a+b)^{-2n-2}abF(n+1, \tfrac{3}{2}, 3, \tfrac{4ab}{(a+b)^2})$$

$$A'' = \tfrac{n(n+1)}{1.2}(a+b)^{-2n-4}aabbF(n+2, \tfrac{5}{2}, 5, \tfrac{4ab}{(a+b)^2})$$

$$A'' = \tfrac{n(n+1)(n+2)}{1.2.3}(a+b)^{-2n-6}a^3b^3F(n+3, \tfrac{7}{2}, 7, \tfrac{4ab}{(a+b)^2})$$

etc.

Fourthly and lastly,

$$A = (a-b)^{-2n}F(n, \tfrac{1}{2}, 1, -\tfrac{4ab}{(a-b)^2})$$

$$A' = n(a-b)^{-2n-2}abF(n+1, \tfrac{3}{2}, 3, -\tfrac{4ab}{(a-b)^2})$$

$$A'' = \tfrac{n(n+1)}{1.2}(a-b)^{-2n-4}aabbF(n+2, \tfrac{5}{2}, 5, -\tfrac{4ab}{(a-b)^2})$$

$$A''' = \tfrac{n(n+1)(n+2)}{1.2.3}(a-b)^{-2n-6}a^3b^3F(n+3, \tfrac{7}{2}, 7, -\tfrac{4ab}{(a-b)^2})$$

etc.

These and the preceding values are easily obtained from

$$\Omega(a+b)^{2n} = (1 - \tfrac{4ab\cos\frac{1}{2}\phi^2}{(a+b)^2})^{-n}$$

$$= 1 + n\tfrac{ab}{(a+b)^2}(r^{\frac{1}{2}}+r^{-\frac{1}{2}})^2 + \tfrac{n(n+1)}{1.2}\cdot\tfrac{aabb}{(a+b)^4}(r^{\frac{1}{2}}+r^{-\frac{1}{2}})^4 + \text{etc.}$$

$$\Omega(a-b)^{2n} = (1 + \tfrac{4ab\sin\frac{1}{2}\phi^2}{(a-b)^2})^{-n}$$

$$= 1 + n\tfrac{ab}{(a-b)^2}(r^{\frac{1}{2}}-r^{-\frac{1}{2}})^2 + \tfrac{n(n+1)}{1.2}\cdot\tfrac{aabb}{(a-b)^4}(r^{\frac{1}{2}}-r^{-\frac{1}{2}})^4 + \text{etc.}$$

FIRST SECTION.

Relations between contiguous functions.

7.

We call *Contiguous Functions* of $F(\alpha, \beta, \gamma, x)$ those functions which arise from it when the first, second or third element is increased or diminished by unity while the other three remain the same. Thus the primary function $F(\alpha, \beta, \gamma, x)$ gives rise to six contiguous functions between any two of which and the primary function we get a very simple linear equation. These equations, fifteen in number, are given below, the fourth element which always remains $= x$ being omitted and the primary function being denoted simply by F, for shortness' sake.

[1] $0 = (\gamma - 2\alpha - (\beta - \alpha)x)F + \alpha(1 - x)F(\alpha + 1, \beta, \gamma) - (\gamma - \alpha)F(\alpha - 1, \beta, \gamma)$

[2] $0 = (\beta - \alpha)F + \alpha F(\alpha + 1, \beta, \gamma) - \beta F(\alpha, \beta + 1, \gamma)$

[3] $0 = (\gamma - \alpha - \beta)F + \alpha(1 - x)F(\alpha + 1, \beta, \gamma) - (\gamma - \beta)F(\alpha, \beta - 1, \gamma)$

[4] $0 = \gamma(\alpha - (\gamma - \beta)x)F - \alpha\gamma(1 - x)F(\alpha + 1, \beta, \gamma) + (\gamma - \alpha)(\gamma - \beta)xF(\alpha, \beta, \gamma + 1)$

[5] $0 = (\gamma - \alpha - 1)F + \alpha F(\alpha + 1, \beta, \gamma) - (\gamma - 1)F(\alpha, \beta, \gamma - 1)$

[6] $0 = (\gamma - \alpha - \beta)F - (\gamma - \alpha)F(\alpha + 1, \beta, \gamma) + \beta(1 - x)F(\alpha, \beta + 1, \gamma)$

[7] $0 = (\beta - \alpha)(1 - x)F - (\gamma - \alpha)F(\alpha - 1, \beta, \gamma) + (\gamma - \beta)F(\alpha, \beta - 1, \gamma)$

[8] $0 = \gamma(1 - x)F - \gamma F(\alpha - 1, \beta, \gamma) + (\gamma - \beta)xF(\alpha, \beta, \gamma + 1)$

[9] $0 = (\alpha - 1 - (\gamma - \beta - 1)x)F + (\gamma - \alpha)F(\alpha - 1, \beta, \gamma) - (\gamma - 1)(1 - x)F(\alpha, \beta, \gamma - 1)$

[10] $0 = (\gamma - 2\beta + (\beta - \alpha)x)F + \beta(1 - x)F(\alpha, \beta + 1, \gamma) - (\gamma - \beta)F(\alpha, \beta - 1, \gamma)$

[11] $0 = \gamma(\beta - (\gamma - \alpha)x)F - \beta\gamma(1 - x)F(\alpha, \beta + 1, \gamma) + (\gamma - \alpha)(\gamma - \beta)xF(\alpha, \beta, \gamma + 1)$

[12] $0 = (\gamma - \beta - 1)F + \beta F(\alpha, \beta + 1, \gamma) - (\gamma - 1)F(\alpha, \beta, \gamma - 1)$

[13] $0 = \gamma(1 - x)F - \gamma F(\alpha, \beta - 1, \gamma) + (\gamma - \alpha)xF(\alpha, \beta, \gamma + 1)$

[14] $0 = (\beta - 1 - (\gamma - \alpha - 1)x)F + (\gamma - \beta)F(\alpha, \beta - 1, \gamma) - (\gamma - 1)(1 - x)F(\alpha, \beta, \gamma - 1)$

[15] $0 = \gamma(\gamma - 1 - (2\gamma - \alpha - \beta - 1)x)F + (\gamma - \alpha)(\gamma - \beta)xF(\alpha, \beta, \gamma + 1) - \gamma(\gamma - 1)(1 - x)F(\alpha, \beta, \gamma - 1)$

8.

These formulae can be demonstrated as follows. If we put

$$\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\beta(\beta+1)\cdots(\beta+m-2)}{1.2.3\cdots m.\gamma(\gamma+1)\cdots(\gamma+m-1)} = M$$

the coefficient of x^m will be

$$\begin{array}{lll} \text{in } F & \cdots\cdots & \alpha(\beta+m+1)M \\ \text{in } F(\alpha,\beta-1,\gamma) & \cdots\cdots & \alpha(\beta-1)M \\ \text{in } F(\alpha+1,\beta,\gamma) & \cdots\cdots & (\alpha+m)(\beta+m-1)M \\ \text{in } F(\alpha,\beta,\gamma-1) & \cdots\cdots & \frac{\alpha(\beta+m-1)(\gamma+m-1)}{\gamma-1}M \end{array}$$

also the coefficient of x^{m-1} in $F(\alpha+1,\beta,\gamma)$ or the coefficient of x^m in $xF(\alpha+1,\beta,\gamma)$

$$= m(\gamma+m-1)M.$$

Hence follows at once the truth of formulae 5 and 3; formula 12 is derived from 5 by interchanging α with β, and then formula 2 by elimination from these two. 6 is derived from 3 by the same interchange; 9 from 6 and 12 combined, from which again 14 is obtained by the interchange; these two combined give us 7, and lastly from 2 and 6 we deduce 1, and thence 10 by the interchange. Formula 8 can be obtained in the same way as formulae 5 and 3 above from the consideration of coefficients, (in fact *all* the fifteen formulae can if we please, be so obtained), but the following method of deriving it from those already known is more elegant. Changing the element α into $\alpha-1$, and γ into $\gamma+1$, in the formula 5, we have

$$0 = (\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1) + (\alpha-1)F(\alpha,\beta,\gamma+1) - \gamma F(\alpha-1,\beta,\gamma)$$

But, by change of γ into $\gamma+1$, 9 becomes

$$0 = (\alpha-1-(\gamma-\beta)x)F(\alpha,\beta,\gamma+1)+(\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1)-\gamma(1-x)F(\alpha,\beta,\gamma)$$

By subtracting the latter from the former, we get 8 and thence 13 by the interchange. From 1 and 8 is derived 4, and thence 11 by the interchange. Lastly 15 is deduced from 8 and 9.

9.

If $\alpha' - \alpha, \beta' - \beta, \gamma' - \gamma$ and also $\alpha'' - \alpha, \beta'' - \beta, \gamma'' - \gamma$ be integers (positive or negative), we can pass from the function $F(\alpha, \beta, \gamma)$ to the function $F(\alpha', \beta', \gamma')$ and from this to the function $F(\alpha'', \beta'', \gamma'')$ by a series of similar functions such that each one is contiguous to the one going before and the one coming after, that is, by first changing one element, say α, successively by unity until from $F(\alpha, \beta, \gamma)$ we arrive at $F(\alpha', \beta, \gamma)$, then changing the second element until we arrive at $F(\alpha', \beta', \gamma)$ and lastly changing the third element until we arrive at $F(\alpha', \beta', \gamma')$, and so on to $F(\alpha'', \beta'', \gamma'')$. Since by Art. 7, there are linear equations between the first, second and third, and generally between any three consecutive functions of this series, it is easy to see that a linear equation between the functions $F(\alpha, \beta, \gamma)$, $F(\alpha', \beta', \gamma')$, $F(\alpha'', \beta'', \gamma'')$ can be deduced from them by elimination, so that generally speaking, from two functions having the first three elements differing by integers we can derive a third function having the same property, the fourth element always remaining unchanged. For the rest, it is sufficient for us to have established this remarkable truth generally, nor need we here dwell upon the simplifications by which the operations to that end can be made as brief as possible.

10.

Suppose, for example, we are given the functions,

$$F(\alpha, \beta, \gamma), F(\alpha+1, \beta+1, \gamma+1), F(\alpha+2, \beta+2, \gamma+2)$$

between which it is required to find a linear equation. Let us connect them by contiguous functions as follows:

$$\begin{aligned}
&F(\alpha, \beta, \gamma) = F \\
&F(\alpha+1, \beta, \gamma) = F' \\
&F(\alpha+1, \beta+1, \gamma) = F'' \\
&F(\alpha+1, \beta+1, \gamma+1) = F''' \\
&F(\alpha+2, \beta+1, \gamma+1) = F'''' \\
&F(\alpha+2, \beta+2, \gamma+1) = F''''' \\
&F(\alpha+2, \beta+2, \gamma+2) = F''''''
\end{aligned}$$

Then we shall have five linear equations (from the formulae 6, 13, 5 Art. 7):

I. $\quad 0 = (\gamma - \alpha - 1)F - (\gamma - \alpha - 1 - \beta)F' - \beta(1-x)F''$

II. $\quad 0 = \gamma F' - \gamma(1-x)F'' - (\gamma - \alpha - 1)xF'''$

III. $\quad 0 = \gamma F'' - (\gamma - \alpha - 1)F''' - (\alpha+1)F''''$

VI. $\quad 0 = (\gamma - \alpha - 1)F''' - (\gamma - \alpha - 2 - \beta)F'''' - (\beta+1)(1-x)F'''''$

V. $\quad 0 = (\gamma+1)F'''' - (\gamma+1)(1-x)F''''' - (\gamma - \alpha - 1)F''''''$

From **I** and **II** , by eliminating F', we get

VI. $$0 = \gamma F - \gamma(1-x)F'' - (\gamma - \alpha - \beta - 1)xF'''$$

From this and **III**, eliminating F''

VII. $$0 = \gamma F - (\gamma - \alpha - 1 - \beta x)F''' - (\alpha + 1)(1-x)F''''$$

Then from **IV** and **V** , eliminating F'''''

VIII. $$0 = (\gamma + 1)F''' - (\gamma + 1)F'''' + (\beta + 1)xF''''''$$

From this and **VII** , eliminating F''''

IX. $$0 = \gamma(\gamma + 1)F - (\gamma + 1)(\gamma - (\alpha + \beta + 1)x)F''' - (\alpha + 1)(\beta + 1)x(1-x)F''''''$$

11.

If we wish to obtain all the relations between the three functions $F(\alpha, \beta, \gamma)$, $F(\alpha + \lambda, \beta + \mu, \gamma + \nu)$, $F(\alpha + \lambda', \beta + \mu', \gamma + \nu')$, in which $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are each equal to 0 or +1 or −1, the number of the formulae will amount to 325. Such a collection would not be useless, at least of the simpler of those formulae; but here let it suffice to put down only a few which can be easily demonstrated either by deduction from the formulae of Art. 7, or if preferred, by the method similar to that by which two of them have been first obtained in Art. 8.

[16] $F(\alpha, \beta, \gamma) - F(\alpha, \beta, \gamma - 1) = -\frac{\alpha\beta x}{\gamma(\gamma-1)} F(\alpha + 1, \beta + 1, \gamma + 1)$

[17] $F(\alpha, \beta + 1, \gamma) - F(\alpha, \beta, \gamma) = \frac{\alpha x}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1)$

[18] $F(\alpha + 1, \beta, \gamma) - F(\alpha, \beta, \gamma) = \frac{\beta x}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1)$

[19] $F(\alpha, \beta + 1, \gamma + 1) - F(\alpha, \beta, \gamma) = \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} F(\alpha + 1, \beta + 1, \gamma + 2)$

[20] $F(\alpha + 1, \beta, \gamma + 1) - F(\alpha, \beta, \gamma) = \frac{\beta(\gamma-\alpha)x}{\gamma(\gamma+1)} F(\alpha + 1, \beta + 1, \gamma + 2)$

[21] $F(\alpha - 1, \beta + 1, \gamma) - F(\alpha, \beta, \gamma) = \frac{(\alpha-\beta-1)x}{\gamma} F(\alpha, \beta + 1, \gamma + 1)$

[22] $F(\alpha+1,\beta-1,\gamma)-F(\alpha,\beta,\gamma)=\frac{(\beta-\alpha-1)x}{\gamma}F(\alpha+1,\beta,\gamma+1)$

[23] $F(\alpha,\beta+1,\gamma)-F(\alpha+1,\beta,\gamma)=\frac{(\alpha-\beta)x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$

SECOND SECTION.

Continued Fractions.

12.

Denoting $\frac{F(\alpha,\beta+1\gamma+1,x)}{F(\alpha,\beta,\gamma,x)}$ by $G(\alpha,\beta,\gamma,x)$, we have

$$\frac{F(\alpha+1,\beta,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \frac{F(\beta,\alpha+1,\gamma+1,x)}{F(\beta,\alpha,\gamma,x)} = G(\beta,\alpha,\gamma,x)$$

and hence dividing the equation 19 by $F(\alpha,\beta+1,\gamma+1,x)$,

$$1 - \frac{1}{G(\alpha,\beta,\gamma,x)} = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} xG(\beta+1,\alpha,\gamma+1,x),$$

or

$$[24]\ \ G(\alpha,\beta,\gamma,x) = \cfrac{1}{1 - \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} xF(\beta+1,\alpha,\gamma+1,x)}$$

and since similarly

$$G(\beta+1,\alpha,\gamma+1,x) = \cfrac{1}{1 - \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)} xG(\alpha+1,\beta+1,\gamma+2,x)}.$$

etc., $G(\alpha,\beta,\gamma,x)$ can be reduced to a continued fraction, thus:

$$[25]\ \ \frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \cfrac{1}{1 - \cfrac{ax}{1 - \cfrac{bx}{1 - \cfrac{cx}{1 - \cfrac{dx}{1 - \text{etc.}}}}}}$$

where

$$\alpha = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} \qquad b = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)}$$

$$c = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)} \qquad d = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)}$$

$$e = \frac{(\alpha+2)(\gamma+2-\beta)}{(\gamma+4)(\gamma+5)} \qquad f = \frac{(\beta+3)(\gamma+3-\alpha)}{(\gamma+5)(\gamma+6)}$$

etc., the law of progression being obvious.

Again from the equations $17, 18, 21, 22$, follow

$$[26]\quad \frac{F(\alpha,\beta+1,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{\alpha x}{\gamma}G(\beta+1,\alpha,\gamma,x)}$$

$$[27]\quad \frac{F(\alpha+1,\beta,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{\beta x}{\gamma}G(\alpha+1,\beta,\gamma,x)}$$

$$[28]\quad \frac{F(\alpha-1,\beta+1,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{(\alpha-\beta-1)x}{\gamma}G(\beta+1,\alpha-1,\gamma,x)}$$

$$[29]\quad \frac{F(\alpha+1,\beta-1,\gamma,x)}{F(\alpha,\beta,\gamma,x)} = \frac{1}{1-\frac{(\beta-\alpha-1)x}{\gamma}G(\alpha+1,\beta-1,\gamma,x)}$$

whence substituting for the function G its values in continued fractions we obtain as many new continued fractions.

For the rest, it is at once evident that the continued fraction in the formula 25 will terminate if any one of the numbers $\alpha, \beta, \gamma-\alpha, \gamma-\beta$ be a negative integer, but otherwise it will run on to infinity.

13.

The continued fractions obtained in the preceding article are of the greatest importance and it may be asserted that there are hardly any continued fractions proceeding according to an obvious law, hitherto obtained by the analysts, which are not contained under ours as special cases. There is a specially noteworthy case when in the formula 25 we put $\beta = 0$ whence $F(\alpha,\beta,\gamma,x) = 1$ so that writing $\gamma - 1$ for γ, we obtain

$$[30]\quad F(\alpha,1,\gamma,x) = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

$$= \cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{1-\text{etc.}}}}}}$$

$$\alpha = \frac{\alpha}{\gamma} \qquad b = \frac{\gamma-\alpha)}{\gamma(\gamma+1)}$$

$$c = \frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)} \qquad d = \frac{2(\gamma+1-\alpha)}{(\gamma+2)(\gamma+3)}$$

$$e = \frac{(\alpha+2)(\gamma+1)}{(\gamma+3)(\gamma+4)} \qquad f = \frac{3(\gamma+2-\alpha)}{(\gamma+4)(\gamma+5)}$$

etc.

14.

It will be worth while to put down here certain special cases. From the formula **I** , Art. 5, putting $t = 1, \beta = 1$, we get

$$[31]\ (1+u)^n = \cfrac{1}{1 - \cfrac{nu}{1 + \cfrac{\frac{n+1}{2}u}{1 - \cfrac{\frac{n-1}{2.3}u}{1 + \cfrac{\frac{2(2n+2)}{3.4}u}{1 - \cfrac{\frac{2(n-2)}{4.5}u}{1 + \text{etc.}}}}}}}$$

From the formulae **VI** ,**VII** , Art. 5, we have

$$[32]\ \log(1+t) = \cfrac{t}{1 + \cfrac{\frac{1}{2}t}{1 + \cfrac{\frac{1}{6}t}{1 + \cfrac{\frac{2}{6}t}{1 + \cfrac{\frac{2}{10}t}{1 + \cfrac{\frac{3}{10}t}{1 + \cfrac{\frac{3}{14}t}{1 + \text{etc.}}}}}}}}$$

$$[33]\ \log\frac{1+t}{1-t} = \cfrac{2t}{1 - \cfrac{\frac{1}{3}tt}{1 - \cfrac{\frac{2.2}{3.5}tt}{1 - \cfrac{\frac{3.3}{5.7}tt}{1 - \cfrac{\frac{4.4}{7.9}tt}{1 - \text{etc.}}}}}}$$

Also we have

$$[34]\ e^t = \cfrac{1}{1 - \cfrac{t}{1 + \cfrac{\frac{1}{2}t}{1 - \cfrac{\frac{1}{6}t}{1 + \cfrac{\frac{1}{6}t}{1 - \cfrac{\frac{1}{10}t}{1 + \cfrac{\frac{1}{10}t}{1 - \text{etc.}}}}}}}}$$

$$[35]\ t = \cfrac{\sin t \cos t}{1 - \cfrac{\frac{1.2}{1.3}\sin t^2}{1 - \cfrac{\frac{1.2}{3.5}\sin t^2}{1 - \cfrac{\frac{3.4}{5.7}\sin t^2}{1 - \cfrac{\frac{3.4}{7.9}\sin t^2}{1 - \cfrac{\frac{5.6}{9.11}\sin t^2}{1 - \text{etc.}}}}}}}$$

If we put, $\alpha = 3, \gamma = \frac{5}{2}$, in the formula 30, the continued fraction in Art. 90 of *Theoria motus corporum coelestium* (Theory of Motion of Heavenly Bodies) follows at one. In the same place two other continued fractions are given, the development of which it may not be out place to add here. Let

$$Q = 1 - \cfrac{\frac{5.8}{7.9}x}{1 - \cfrac{\frac{1.4}{9.11}x}{1 - \frac{7.10}{11.13}x \text{ etc.,}}}$$

then (1. c.) $\quad x - \xi = \dfrac{x}{1 + \frac{2x}{35Q}} = \dfrac{xQ}{Q + \frac{2}{35}x}$, so that

$$\xi = \frac{\frac{2}{35}xx}{Q + \frac{2}{35}x}$$

which is the first formula: the next is found in the following manner.

Let

$$Q = 1 - \cfrac{\frac{1.4}{7.9}x}{1 - \cfrac{\frac{5.8}{9.11}x}{1 - \cfrac{\frac{3.6}{11.13}x}{1 - \frac{7.10}{13.15}x \text{ etc.,}}}}$$

then, by formula 25,

$$\frac{1}{R} = G(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x), \text{ and } \frac{1}{Q} = G(\frac{5}{2}, -\frac{1}{2}, \frac{7}{2}, x)$$

Hence

$$RF(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x) = F(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x)$$

$$QF(\frac{5}{2}, \frac{1}{2}, \frac{9}{2}, x) = F(\frac{5}{2}, -\frac{1}{2}, \frac{7}{2}, x)$$

or with the first and second elements interchanged

$$QF(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x) = F(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x)$$

But by equation 21, we have

$$F(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x) - F(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x) = -\frac{4}{7}xF(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x)$$

whence $Q = R - \frac{4}{7}x$; substituting this value in the formula given above we get

$$\xi = \frac{\frac{2}{35}xx}{R - \frac{18}{35}x}$$

which is the second formula.

Putting in the formula 30, $\alpha = \dfrac{m}{n}$, $x = -\gamma nt$, we shall have, for an infinitely great value of γ

$$[36]\quad F(\frac{m}{n}, 1, \gamma, -\gamma nt) = 1 - mt + m(m+n)tt - m(m+n)(m+2n)t^3 + \text{ etc.}$$

$$= \cfrac{1}{1 + \cfrac{mt}{1 + \cfrac{nt}{1 + \cfrac{(m+n)t}{1 + \cfrac{2nt}{1 + \cfrac{(m+2n)t}{1 + 3nt \text{ etc.}}}}}}}$$

THIRD SECTION.

Of the sum of our series, when the fourth element is put $= 1$, with the discussion of certain other transcendental functions.

15.

As long as all the three elements α, β, γ are positive quantities, the coefficients of the powers of the fourth element x are all positive; when however one or other of these elements is negative, the coefficients will all be affected with the same sign at all events after a certain power x^m if only m be taken greater than the absolute value of the greatest negative element. Moreover it is at once evident from this, that the sum of the series for $x = 1$ can not be finite unless the coefficients at al events after a certain term diminish indefinitely, or, to speak according to the manner of analysts, unless the coefficients of the term $x^\infty = 0$. But for the sake of those who are in favour of the rigorous methods of the ancient geometers, we shall show with all the rigour,

1stly, that the coefficients (unless indeed the series terminates) will increase indefinitely if $\alpha + \beta - \gamma - 1$ be a positive quantity.

2ndly, that the coefficients will tend continually towards a finite limit if $\alpha + \beta - \gamma - 1 = 0$.

3rdly, that the coefficients will decrease indefinitely if $\alpha + \beta - \gamma - 1$ be a negative quantity.

4thly, that the sum of our series for $x = 1$ will be infinite, notwithstanding the convergency in the third case, if $\alpha + \beta - \gamma$ be either positive or $= 0$.

5thly, that the sum will be *finite* , if $\alpha + \beta - \gamma$ be a negative quantity.

16.

We shall apply this discussion more generally to the infinite series M, M', M'', M'''&c., so formed that the quotients $\frac{M'}{M}, \frac{M''}{M'}, \frac{M'''}{M''}$, etc. are the respective values of the fraction

$$\frac{t^\lambda + At^{\lambda-1} + Bt^{\lambda-2} + Ct^{\lambda-3} + \text{etc.}}{t^\lambda + at^{\lambda-1} + bt^{\lambda-2} + ct^{\lambda-3} + \text{etc.}}$$

for $t = m, t = m + 1, t = m + 2$, etc. For shortness' sake, we shalldenote the numerator of this fraction by P, the denominator by p; moreover we shall suppose that P and p are not identical, that is, that the differences $A - a, B - b, C - c$, etc. do not all vanish at the same time.

I. When of the differences $A - a, B - b, C - c$, etc., the first that does not vanish is positive, a certain limit l can be assigned such that as soon as the value of t shall exceed this limit, the values of functions P and p will always certainly become positive and $P > p$. Clearly this will be the case, if for l be taken the greatest real root at all; the above mentioned property will hold for any value whatsoever of t. Hence in the series $\frac{M'}{M}, \frac{M''}{M'}, \frac{M'''}{M''}$, etc., (if not from the beginning,) at all events after a certain interval, all the terms are positive and greater than unity; if then none of them either = 0 or become infinitely great, it is clear, that

The series M, M', M'', M''', etc. will have all its terms, if not from the beginning, at all events after a certain interval, affected with the same sign and continually increasing.

For the same reason, if of the differences $A - a, B - b, C - c$, etc., the first that does not vanish is negative, the series M, M', M'', M''', etc. will all its terms, if not from the beginning, at least after a certain interval, affected with the same sign, and continually decreasing.

II. If now the coefficients A, a are unequal, the terms of the series M, M', M'', M''' will either increase beyond all limits or diminish indefinitely according as the difference $A - a$ is positive or negative: this may be demonstrated thus. If $A - a$ be positive, take an integer h, such that $h(A - a) > 1$ and put $\frac{M^h}{m} = N, \frac{M'^h}{m+1} = N', \frac{M''^h}{m+2} = N'', \frac{M'''^h}{m+3} = N'''$ etc., and also $tP^h = Q, (t+1)p^h = q$. Then evidently $\frac{N'}{N}, \frac{N''}{N'}, \frac{N'''}{N''}$, etc. are the values of fraction $\frac{Q}{q}$ when t is put $= m, m+1, m+2$, etc. successively, and Q, q are algebraic functions of the form

$$Q = t^{\lambda h+1} + hAt^{\lambda h} + \text{etc.}$$

$$q = t^{\lambda h+1} + (ha+1)t^{\lambda h} + \text{etc.}$$

wherefore since by hypothesis the difference $hA - (ha + 1)$ is positive quantity, terms of the series N, N', N'', N''', etc. will if not from the beginning, certainly after a certain interval, increase continually (by **I**): hence the terms of the series $mN, (m+1)N', (m+2)N'', (m+3)N'''$, etc. will necessarily increase beyond all limits, and hence also terms of the series M, M', M'', M''', etc., as these raised to the power h are equal to the former.

If $A - a$ be negative, integer h should be taken such that $h(a - A)$ shall be greater than 1, whence by reasoning similar to the above, terms of the series $mM^h, (m+1)M'^h, (m+2)M''^h, (m+3)M'''^h$, etc. will continually decrease after a certain interval. Therefore the terms of the series M^h, M'^h, M''^h etc. as also the terms of the series M, M', M'', M''' etc. will necessarily decrease indefinitely.

III. If however the first coefficients A, a are equal, terms of the series M, M', M'', M''' etc. will continually converge towards a finite limit; which may be demonstrated thus. Suppose first that the terms of the series continually increase after a certain interval, or what is the same thing, that of the differences $B - b, C - c$, etc., the first that does not vanish is positive. Let h be an integer such that $h + b - B$ is positive, and put

$$M(\frac{m}{m-1})^h = N, M'(\frac{m+1}{m})^h = N', M''(\frac{m+2}{m+1})^h = N'', \text{etc.}$$

and $(tt-1)^h P = Q, t^{2h} p = q$, so that $\frac{N'}{N}, \frac{N''}{N'}, \frac{N'''}{N''}$, etc. are the values of the fraction $\dfrac{Q}{q}$ when t is put $= m, m+1$, etc. successively.

Since we have

$$Q = t^{\lambda+2h} + At^{\lambda+2h-1} + (B-h)t^{\lambda+2h-2} + \text{etc.}$$

$$q = t^{\lambda+2h} + At^{\lambda+2h-1} + bt^{\lambda+2h-2}, \text{ etc.,}$$

and since by hypothesis $B-h-b$ is negative, the terms of the series N, N', N'', N''' etc. will continually diminish at least after a certain interval, so that the terms of the series M, M', M'', M''' which are always respectively less than those of the former, while continually increasing, can only converge towards a fixed limit.

If the terms of the series M, M', M'', M''' etc. continually diminish after a certain interval, we ought to take h an integer such that $h+B-b$ is a positive quantity, and it can be shown by a precisely similar reasoning that the terms of the series

$$M(\frac{m-1}{m})^h, M'(\frac{m}{m+1})^h, M''(\frac{m+1}{m+2})^h \text{ etc.}$$

continually increase after a certain interval; whence the terms of the series M, M', M'' etc., which are always respectively greater than them, while continually decreasing, can necessarily only decrease towards a finite limit.

IV. Lastly as regards the *sum* of the series whose terms are M, M', M'', M''', etc. in the case when these diminish indefinitely, let us suppose, first, $A-a$ to fall between 0 and -1, that is, $A+1-a$ to be either positive or -0. Let h be a positive integer, arbitrary in the case when $A+1-a$ is a positive quantity or such as to make the quantity $h+m+A+B-b$ positive in the case when $A+1-a=0$. Then we shall have

$$P(t-(m+h-1)) = t^{\lambda+1} + (A+1-m-h)t^{\lambda} + (B-A(m+h-1))t^{\lambda-1} \text{ etc.}$$

$$p(t-(m+h)) = t^{\lambda+1} - (a-m-h)t^{\lambda} + (b-a(m+h))t^{\lambda-1} \text{ etc.,}$$

where either $A+1-m-h-(a-m-h)$ will be positive, or, if this should become $=0$, $B-A(m+h-1)-(b-a(m+h))$ at least will be positive. Hence by I , a certain value l can be assigned to the quantity t, such that when it exceeds that value, the values of the fraction $\dfrac{P(t-(m+h-1))}{p(t-(m+h))}$ become always positive and greater than unity. Let n be an integer greater than l and also greater than h; and let the terms of the series M, M', M'', M''' etc. which correspond to the values $t=m+n$, $t=m+n+1$, $t=m+n+2$, etc., be N, N', N'', N''', etc. Then

$$\frac{(n+1-h)N'}{(n-h)N}, \frac{(n+2-h)N''}{(n+1-h)N'}, \frac{(n+3-h)N'''}{(n+2-h)N''}, \text{ etc.}$$

will be positive quantities greater than unity, whence

$$N' > \frac{(n-h)N}{n+1-h}, N'' > \frac{(n-h)N}{n+2-h}, N''' > \frac{(n-h)N}{n+3-h}, \text{ etc.}$$

so that the sum of the series $N + N' + N'' + N'''+$ etc. will be greater than the sum of the series

$$(n-h)N\left(\frac{1}{n-h} + \frac{1}{n+1-h} + \frac{1}{n+2-h} + \frac{1}{n+3-h} + \text{ etc.}\right)$$

whatever the number of terms taken. But the latter series will, when the number of terms is infinite, exceed all limits, since the sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}+$ etc. which is known to be infinite, remains infinite also in the terms $1 + \frac{1}{2} + \frac{1}{3} +$ etc. $+ \dfrac{1}{n-1-h}$ at the begining be omitted. Hence the sum of the series $N + N' + N'' + N'''+$ etc. and consequently also the sum of the series $M + M' + M'' + M'''+$ etc., of which the former is a part will increase beyond all limits.

V. When however $A - a$ is a negative quantity whose absolute value is greater than unity, the sum of the infinite series $M + M' + M'' + M'''+$ etc. will certainly be finite. For, let h be a positive quantity less than $a - A - 1$, then it can be demonstrated by a similar reasoning that a certain value l can be assigned to the quantity t, beyond which the fraction $\dfrac{Pt}{p(t-h-1)}$ will always have positive values less than unity. If therefore for n be taken an integer greater than l, m, or $h + 1$ and if the terms of the series M, M', M'', M''', etc. corresponding to the values $t = n$, $t = n + 1$, $t = n + 2$, etc. be denoted by N, N', N'', etc., we shall have

$$N' < \frac{n-h-1}{n}.N, N'' < \frac{(n-h-1)(n-h)}{n(n+1)}.N, \text{ etc.}$$

so that the sum of the series $N + N' + N''+$ etc. will be, whatever the number of terms taken, less than the product of N into the sum of the same number of terms of the series

$$1 + \frac{n-h-1}{n} + \frac{(n-h-1)(n-h)}{n(n+1)} + \frac{(n-h-1)(n-h)(n-h+1)}{n(n+1)(n+2)} \text{ etc.}$$

Now the sum of this last for any number of terms can be easily found: thus

the first term $= \dfrac{n-1}{h} - \dfrac{n-h-1}{h}$

the sum of the 1st two terms $= \dfrac{n-1}{h} - \dfrac{(n-h-1)(n-h)}{hn}$

the sum of the 1st three terms $= \dfrac{n-1}{h} - \dfrac{(n-h-1)(n-h)(n-h+1)}{hn(n+1)}$ etc.

And since the second part (by **II**) forms a series decreasing beyond all limits, the sum of the series continued to infinity must be put $= \dfrac{n-1}{h}$. Hence $N + N' + N''+$ etc. *ad infinitum* will always remain less than $N.\dfrac{n-1}{h}$ and therefore $M + M' + M''+$ etc. will certainly converge to a finite limit.

VI. What we have demonstrated generally about the series $M, M'M'', M'''$ etc. will be seen to apply to the coefficients of the powers x^m, x^{m+1}, x^{m+2} etc. in the series $F(\alpha, \beta, \gamma, x)$, by putting $\gamma = 2$, $A = \alpha + \beta$, $B = \alpha\beta$, $a = \gamma + 1$, $b = \gamma$; whence the five propositions in the preceding Art. follow at once.

17.

Thus the inquiry about the nature of the sum of the series $F(\alpha, \beta, \gamma, 1)$ is restricted by its nature to the case where $\gamma - \alpha - \beta$ is a positive quantity, when that sum will always be a finite quantity. Also we make the following preliminary observation. If, after a certain term, the coefficients of the series $1 + ax + bxx + cx^3 +$ & c. $= S$ diminish beyond all limits, the product

$$(1 - x)S = 1 + (a - 1)x + (b - a)xx + (c - b)x^3 + \text{ \& c.}$$

ought to become $= 0$ for $x = 1$, even if the sum of the series itself become infinitely great. For since the sum of the first two terms $= a$, of the first three terms $= b$, of the first four terms $= c$, and so on, the limit of the sum of infinite number of terms must $= 0$. When therefore $\gamma - \alpha - \beta$ is a positive quantity, we must have $(1 - x)F(\alpha, \beta, \gamma - 1, x) = 0$, for $x = 1$ whence by equation 15 Art. 7 we shall have

$$0 = \gamma(\alpha + \beta - \gamma)F(\alpha, \beta, \gamma, 1) + (\gamma - \alpha)(\gamma - \beta)F(\alpha, \beta, \gamma + 1, 1), \text{ or}$$

$$[37]\ F(\alpha, \beta, \gamma, 1) = \frac{(\gamma - \alpha)(\gamma - \beta)}{(\gamma + 1)(\gamma - \alpha - \beta)} F(\alpha, \beta, \gamma + 1, 1)$$

And since we get from this successively

$$F(\alpha, \beta, \gamma + 1, 1) = \frac{(\gamma + 1 - \alpha)(\gamma + 1 - \beta)}{(\gamma + 1)(\gamma + 1 - \alpha - \beta)} F(\alpha, \beta, \gamma + 2, 1)$$

$$F(\alpha, \beta, \gamma + 2, 1) = \frac{(\gamma + 2 - \alpha)(\gamma + 2 - \beta)}{(\gamma + 2)(\gamma + 2 - \alpha - \beta)} F(\alpha, \beta, \gamma + 3, 1)$$

and so on, generally, k denoting any positive integer,

$$F(\alpha, \beta, \gamma, 1) = \text{ product of } F(\alpha, \beta, \gamma + k, 1)$$

$$\begin{array}{ll} \text{into} & (\gamma - \alpha)(\gamma + 1 - \alpha)(\gamma + 2 - \alpha) \ldots (\gamma + k - 1 - \alpha) \\ \text{into} & (\gamma - \beta)(\gamma + 1 - \beta)(\gamma + 2 - \beta) \ldots (\gamma + k - 1 - \beta) \end{array}$$

divided by the product of

$$\begin{array}{ll} & \gamma(\gamma + 1)(\gamma + 2) \ldots (\gamma + k - 1) \\ \text{into} & (\gamma - \alpha - \beta)(\gamma + 1 - \alpha - \beta)(\gamma + 2 - \alpha - \beta) \ldots ((\gamma + k - 1 - \alpha - \beta) \end{array}$$

18.

We shall hereafter introduce the following notation:-

$$[38]\ \prod(k,z) = \frac{1.2.3\ldots k}{(z+1)(z+2)(z+3)\ldots(z+k)}k^z$$

where k is by its nature understood to be a positive integer, with which restriction $\prod(k,z)$ stands for a completely determined function of two quantities k, z. Then it is easily seen that the theorem at the end of the preceding Art. can be put thus:

$$[39]\ F(\alpha,\beta,\gamma,1) = \frac{\prod(k,\gamma-1).\prod(k,\gamma-\alpha-\beta-1)}{\prod(k,\gamma-\alpha-1).\prod(k,\gamma-\beta-1)}.F(\alpha,\beta,\gamma+k,1).$$

19.

It will be worth while to examine more closely the nature of the function $\prod(k,z)$. When z is a negative integer, the function will evidently acquire an infinitely great value if we give a sufficiently great value to k. For values of z not negative, however, we have

$$\prod(k,0) = 1$$

$$\prod(k,1) = \frac{1}{1+\frac{1}{k}}$$

$$\prod(k,2) = \frac{1.2}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)}$$

$$\prod(k,3) = \frac{1.2.3}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)},$$

etc., generally

$$[40]\ \prod(k,z) = \frac{1.2.3\ldots z}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)\cdots\left(1+\frac{z}{k}\right)}$$

Also generally for *any* value *whatsoever* of z, we have

[41] $\prod(k, z+1) = \prod(k, z).\dfrac{1+z}{1+\dfrac{1+z}{k}}$

[42] $\prod(k+1, z) = \prod(k, z).\left(\dfrac{\left(1+\dfrac{1}{k}\right)^{z+1}}{1+\dfrac{1+z}{k}}\right)$

so that since $\prod(1, z) = \dfrac{1}{z+1}$,

[43] $\prod(k, z) = \dfrac{1}{z+1}\cdot\dfrac{2^{z+1}}{1^z.(2+z)}\cdot\dfrac{3^{z+1}}{2^z.(3+z)}\cdot\dfrac{4^{z+1}}{3^z.(4+z)}\cdots\dfrac{k^{z+1}}{(k-1)^z.(k+z)}$

20.

But the *limit* to which for a given value of z the function $\prod(k, z)$ continually converges when k increases indefinitely is especially worthy of notice. First let h be a finite value of k greater than z; then it is evident that if k be supposed to increase from h to $h+1$, the logarithm of $\prod(k, z)$ will also receive an increment which is expressed by the following convergent series

$$\frac{z(1+z)}{2(h+1)^2} + \frac{z(1-zz)}{3(h+1)^3} + \frac{z(1+z^3)}{4(h+1)^4} + \frac{z(1-z^4)}{5(h+1)^5} + \text{ etc.}$$

If therefore k increases from h to $h+n$, the logarithm of $\prod(k, z)$ will receive the increment

$$\tfrac{1}{2}z(1+z)\left[\frac{1}{(h+1)^2} + \frac{1}{(h+2)^2} + \frac{1}{(h+3)^2} + \cdots + \frac{1}{(h+n)^2}\right]$$

$$\tfrac{1}{3}z(1-zz)\left[\frac{1}{(h+1)^3} + \frac{1}{(h+2)^3} + \frac{1}{(h+3)^3} + \cdots + \frac{1}{(h+n)^3}\right]$$

$$\tfrac{1}{4}z(1+z^3)\left[\frac{1}{(h+1)^4} + \frac{1}{(h+2)^4} + \frac{1}{(h+3)^4} + \cdots + \frac{1}{(h+n)^4}\right]$$

$$+ \text{ etc.}$$

which can easily be shown to remain always finite, even if n increases to infinity. Wherefore, unless there be an infinite factor already in $\prod(k, z)$, i.e., unless z be a negative integer, the limit of $\prod(k, z)$ for $k = \infty$ will certainly be a finite quantity. Clearly therefore $\prod(\infty, z)$ depends solely upon z or is a function entirely determined by z, so that we shall denote it simply as $\prod z$. Thus we define $\prod z$ as the value of

$$\frac{1.2.3\ldots k.k^z}{(z+1)(z+2)\cdots(z+k)}$$

for $k=\infty$ or, if you prefer it, as the limit of the infinite product

$$\frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z(2+z)}\cdot\frac{3^{z+1}}{2^z(3+z)}\cdot\frac{4^{z+1}}{3^z(4+z)}\cdot\text{ etc.}$$

21.

Equation 41 leads at once to the fundamental equation

[44] $\prod(z+1)=(z+1)\prod z$

whence generally, n denoting any positive integer

[45] $\prod(z+n)=(z+1)(z+2)(z+3)\ldots(z+n)\prod z$

For any negative integral value of z, the value of the function $\prod z$ will be infinitely great; for values, integral and not negative, we have

$\prod(0)=1$

$\prod(1)=1$

$\prod(2)=2$

$\prod(3)=6$

$\prod(4)=24$ etc.

and generally

[46] $\prod z=1.2.3\ldots z.$

But this property of our function can scarcely be offered as its definition, as it is restricted by its nature to the integral values, and is common to an infinite number of other series besides our function (e.g. $\cos 2\pi z.\prod z$, $\cos 2\pi z^{2n}.\prod z$ etc. where π stands for the semi-circumference of the circle whose radius $=1$).

22.

The function $\prod(k,z)$, although apparently more general than $\prod(z)$, will henceforth be superfluous, since it can easily be expressed in terms of the latter. For from the equations 38, 45, 46 combined we get

[47] $\prod(k,z) = \dfrac{k^z \prod k. \prod z}{\prod(k+z)}$

Again the connexion of these functions with what Kramp has called *facultates numericae* , is at once evident. That is, the *facultates numerica* which this author denotes by a^{bIc}, is in our notation

$$= \frac{c^b b^{\frac{a}{c}-1} \prod b}{\prod\left(b, \dfrac{a}{c}-1\right)} = \frac{c^b \prod\left(\dfrac{a}{c}+b-1\right)}{\prod\left(\dfrac{a}{c}-1\right)}$$

But it seems preferable to introduce a function of *one* variable into analysis rather than a function of three variables, especially when the latter can be reduced to the former.

23.

The continuity of the function $\prod z$ is broken when its value becomes infinitely great, i.e., for negative integral values of z. Thus it will be positive from $z = -1$ up to $z = \infty$ and since at either limit $\prod z$ is infinitely great, there will be an intermediate minimum value which we find to be $= 0.8856024$ and to correspond to the value of the function $\prod z$ becomes negative; between $z = -2$ and $z = -3$, again positive; and so on, as follows at once from equation 44. Moreover it is evident that if all the values of the function $\prod z$ between arbitrary limits differing by unity e.g. from $z = 0$ to $z = 1$ be known, the value of the function for any other real value of z can easily be deduced therefrom by the help of the equation 45. For this purpose, we have constructed the *table* annexed to the end of this section which gives to twenty figures the logarithms of the function $\prod z$, computed from $z = 0$ to $z = 1$, where however it must be born in mind that the last or twentieth figure may be in error by one or two units.

24.

Since the limit of the function $F(\alpha, \beta, \gamma + k, 1)$ when k increases infinitely, is manifestly unity, equation 39 becomes

[48] $F(\alpha, \beta, \gamma, 1) = \dfrac{\prod(\gamma-1). \prod(\gamma-\alpha-\beta-1)}{\prod(\gamma-\alpha-1). \prod(\gamma-\beta-1)}$

which formula presents a complete solution of the question which constitutes the object of this section. From it the following elegant equations are derived at once:

[49] $F(\alpha, \beta, \gamma, 1) = F(-\alpha, -\beta, \gamma - \alpha - \beta, 1)$

[50] $F(\alpha, \beta, \gamma, 1) = F(-\alpha, \beta, \gamma - \alpha, 1) = 1$

[51] $F(\alpha, \beta, \gamma, 1) = F(\alpha, -\beta, \gamma - \beta, 1)$

γ in the first, $\gamma - \beta$ in the second and $\gamma - \alpha$ in the third being necessarily positive.

25.

We shall apply formula 48 to some of the equations of Art. 5 Formula **XIII**. By putting $t = 90^o = \frac{1}{2}\pi$, we have $\frac{1}{2}\pi = F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1)$ which is equivalent to the well known equation

$$\frac{1}{2}\pi = 1 + \frac{1.1}{2.3} + \frac{1.1.3}{2.4.5} + \frac{1.1.3.5}{2.4.6.7} + \text{ etc.}$$

Whence since by formula 48, $F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1) = \frac{\prod \frac{1}{2} . \prod (-\frac{1}{2})}{\prod 0 . \prod 0}$, and since

$$\prod 0 = 1, \prod \frac{1}{2} = \frac{1}{2} \prod \left(-\frac{1}{2}\right), \text{ we have } \pi = \left(\prod \left(-\frac{1}{2}\right)\right)^2 \text{ or}$$

[52] $\prod (-\frac{1}{2}) = \sqrt{\pi}$

[53] $\prod \frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

Formula **XVI** of Art. 5 which is equivalent to the well known equation

$$\sin nt = n \sin t - \frac{n(nn-1)}{2.3} \sin t^3 + \frac{n(nn-1)(nn-9)}{2.3.4.5} \sin t^5 - \text{ etc.}$$

and holds generally for any value of n provided t lies between the limits -90^o and $+90^o$, gives for $t = \frac{1}{2}\pi$,

$$\sin \frac{n\pi}{2} = \frac{n \prod \frac{1}{2} . \prod (-\frac{1}{2})}{\prod (-\frac{1}{2}n) . \prod \frac{1}{2}n}$$

whence is deduced the elegant formula

$$\prod \frac{1}{2}n . \prod \left(-\frac{1}{2}n\right) = \frac{\frac{1}{2}n\pi}{\sin \frac{1}{2}n\pi}, \text{ or putting } n = 2z,$$

[54] $\prod (-z) . \prod (+z) = \dfrac{z\pi}{\sin z\pi}$

[55] $\prod(-z).\prod(z-1) = \dfrac{\pi}{\sin z\pi}$

and putting $z + \frac{1}{2}$ for z

[56] $\prod\left(-\frac{1}{2}+z\right).\prod\left(-\frac{1}{2}-z\right) = \dfrac{\pi}{\cos z\pi}$

From formula 54, with the definition of the function $\prod$, it follows that $\dfrac{z\pi}{\sin z\pi}$ is the limit of the infinite product

$$\frac{(1.2.3.4.\ldots.k)^2}{(1-zz)(4-zz)(9-zz)\ldots(kk-zz)}$$

where k increases indefinitely; and therefore

$$\sin z\pi = z\pi(1-zz)\left(1-\frac{zz}{4}\right)\left(1-\frac{zz}{9}\right) \text{ etc., ad. inf.}$$

and similarly from 56

$$\cos z\pi = (1-4zz)\left(1-\frac{4zz}{9}\right)\left(1-\frac{4zz}{25}\right) \text{ etc., ad. inf.}$$

well known formulae which used to be obtained by analysts by entirely different methods.

26.

The value of the expression

$$\frac{n^{nz}\prod(k,z).\prod\left(k,z-\frac{1}{n}\right).\prod\left(k,z-\frac{2}{n}\right)\ldots.\prod\left(k,z-\frac{n-1}{n}\right)}{\prod(nk,nz)}$$

where n is a positive integer becomes, when properly taken,

$$\frac{(1.2.3.\ldots.k)^n n^{nk}}{1.2.3.\ldots nk.k^{\frac{1}{2}(n-1)}}$$

so that it is independent of z or remains the same whatever be the value of z. It may therefore, since $\prod(k,0) = \prod(nk,0) = 1$, be represented by the product

$$\prod\left(k,-\frac{1}{n}\right).\prod\left(k,-\frac{2}{n}\right).\prod\left(k,-\frac{3}{n}\right)\ldots.\prod\left(k,-\frac{n-1}{n}\right)$$

If then k increases to infinity, we obtain

$$\frac{n^{nz}\prod z\prod\left(z-\frac{1}{n}\right).\prod\left(z-\frac{2}{n}\right).\prod\left(z-\frac{3}{n}\right)\ldots.\prod\left(z-\frac{n-1}{n}\right)}{\prod nz}$$

$$=\prod\left(-\frac{1}{n}\right)\cdot\prod\left(-\frac{2}{n}\right)\cdot\prod\left(-\frac{3}{n}\right)\cdots\cdots\prod\left(-\frac{n-1}{n}\right)$$

The product on the right hand side multiplied by itself with the order of the factors reversed will, by 55, give

$$\frac{\pi}{\sin\frac{1}{n}\pi}\cdot\frac{\pi}{\sin\frac{2}{n}\pi}\cdot\frac{\pi}{\sin\frac{3}{n}\pi}\cdots\cdot\frac{\pi}{\sin\frac{n-1}{n}\pi}=\frac{(2\pi)^{n-1}}{n}$$

whence we have the following elegant theorem

$$[57]\quad \frac{n^{nz}\prod z\prod\left(z-\frac{1}{n}\right)\cdot\prod\left(z-\frac{2}{n}\right)\cdot\prod\left(z-\frac{3}{n}\right)\cdots\cdots\prod\left(z-\frac{n-1}{n}\right)}{\prod nz}=\frac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}}$$

27.

Integral $\int x^{\lambda-1}(1-x^{\mu})^{\nu}dx$ so taken that it vanishes for $x=0$, is expressed by the following series, if λ, μ are positive quantities:

$$\frac{x^{\lambda}}{\lambda}-\frac{\nu x^{\mu+\lambda}}{\mu+\lambda}+\frac{\nu(\nu-1)x^{2\mu+\lambda}}{1.2.(2\mu+\lambda)}-\text{ etc. }=\frac{x^{\lambda}}{\lambda}F\left(-\nu,\frac{\lambda}{\mu},\frac{\lambda}{\mu}+1,x^{\mu}\right)$$

The value of this for $x=1$ will be

$$=\frac{\prod\left(\frac{\lambda}{\mu}\right)\cdot\prod(\nu)}{\lambda\prod\left(\frac{\lambda}{\mu}+\nu\right)}$$

From this theorem, all the relations which Euler has obtained with great labour follow at once. Thus, e.g., putting

$$\int\frac{dx}{\sqrt{(1-x^4)}}=A,\quad\int\frac{xxdx}{\sqrt{(1-x^4)}}=B$$

we shall have $A=\dfrac{\prod\left(\frac{1}{4}\right)\prod\left(-\frac{1}{2}\right)}{\prod\left(-\frac{1}{4}\right)}$, $B=\dfrac{\prod\left(\frac{3}{4}\right)\prod\left(-\frac{1}{2}\right)}{3\prod\left(\frac{1}{4}\right)}=\dfrac{\prod\left(-\frac{1}{4}\right)\prod\left(-\frac{1}{2}\right)}{\prod\left(4\frac{1}{4}\right)}$ so that $AB=\frac{1}{4}\pi$. From this, since $\prod\frac{1}{4}\cdot\prod\left(-\frac{1}{4}\right)=\dfrac{\frac{1}{4}\pi}{\sin\frac{1}{4}\pi}=\dfrac{\pi}{\sqrt{8}}$, we have

$$\prod\tfrac{1}{4}=\sqrt[4]{\tfrac{1}{8}\pi AA}=\sqrt[4]{\tfrac{\pi^3}{128BB}},\quad\prod\left(-\tfrac{1}{4}\right)=\sqrt[4]{\tfrac{\pi^3}{8AA}}=\sqrt[4]{2\pi BB}$$

The numerical value of A as computed by Stirling is 1.31102877714605987; the value of B according to the same author $=0.59907011736779611$, according to our calculation depending upon our own method

$$=0.59907011736779610372.$$

It can be easily demonstrated generally that the value of the function $\prod z$, z being a rational quantity $= \dfrac{m}{\mu}$ where m, μ are integers, can be deduced from $\mu - 1$ determinate values of such integrals for $x = 1$, and that too by very many different methods. for taking λ as an integer, and ν as a fraction with the denominator $= \mu$, the value of that integral is always reduced to three $\prod z$, where z is a fraction with the denominator $= \mu$; but any $\prod z$ of this kind can be reduced to $\prod\left(-\dfrac{1}{\mu}\right)$ or to $\prod\left(-\dfrac{2}{\mu}\right)$ or to $\prod\left(-\dfrac{3}{\mu}\right)$ and so on or to $\prod\left(-\dfrac{\mu-1}{\mu}\right)$ by formula 45 if z be really a fraction; if z be an integer, $\prod z$ is known by itself. But from those values of the integrals, generally speaking, any $\prod\left(-\dfrac{m}{\mu}\right)$ can be obtained by elimination if $m < \mu$.[2] Indeed only half the number of such integrals are sufficient if we have recourse to formula 54 as well. Thus e.g. putting

$$\int \frac{dx}{\sqrt[5]{(1-x^5)}} = C, \quad \int \frac{dx}{\sqrt[5]{(1-x^5)^2}} = D, \quad \int \frac{dx}{\sqrt[5]{(1-x^5)^3}} = E,$$

$$\int \frac{dx}{\sqrt[5]{(1-x^5)^4}} = F,$$

$$C = \prod \tfrac{1}{5} \cdot \prod(-\tfrac{1}{5}), \quad D = \frac{\prod \frac{1}{5} \cdot \prod(-\frac{2}{5})}{\prod(-\frac{1}{5})}$$

$$E = \frac{\prod \frac{1}{5} \cdot \prod(-\frac{3}{5})}{\prod(-\frac{2}{5})}, \quad F = \frac{\prod \frac{1}{5} \cdot \prod(-\frac{4}{5})}{\prod(-\frac{3}{5})}$$

Hence since $\prod \frac{1}{5} = \frac{1}{5} \prod(-\frac{4}{5})$, we have

$$\prod(-\tfrac{1}{5}) = \sqrt[5]{\frac{5C^4}{DEF}}, \quad \prod(-\tfrac{2}{5}) = \sqrt[5]{\frac{25C^3D^3}{EEFF}}, \quad \prod(-\tfrac{3}{5}) = \sqrt[5]{\frac{125CCDDEE}{F^3}},$$

$$\prod(-\tfrac{4}{5}) = \sqrt[5]{625CDEF}$$

Formulae 54, 55 give in addition to these

$$C = \frac{\pi}{\sin\frac{1}{5}\pi}, \quad \frac{D}{F} = \frac{\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}$$

so that two integrals D and E or E and F are sufficient for the computing of all the values $\prod(-\frac{1}{5})$, $\prod(-\frac{2}{5})$ etc.

28.

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[2] This elimination will be effected by means of linear equations only if we substitute for the quantities their logarithms.

Putting $y = \nu x$ and $\mu = 1$, $\dfrac{\prod \lambda . \prod \nu}{\lambda \prod (\lambda + \nu)}$ will be the value of $\int \dfrac{y^{\lambda-1}(1 - \frac{y}{\nu})^\nu dy}{\nu^\lambda}$ from $y = 0$ to $y = \nu$, or the value of the integral $\int y^{\lambda-1}(1 - \frac{y}{\nu})^\nu dy$ between these limits $= \dfrac{\nu^\lambda \prod \lambda . \prod \nu}{\lambda \prod (\lambda + \nu)} = \dfrac{\prod (\nu, \lambda)}{\lambda}$ (formula 47) provided ν denotes an integer. Now, with ν increasing to infinity the limit of $\prod (\nu, \lambda)$ will be $\prod \lambda$ and the limit of $(1 - \frac{y}{\nu})^\nu$ will be e^{-y}, where e denotes the base of the hyperbolic logarithms. Wherefore if λ is positive, $\dfrac{\prod \lambda}{\lambda}$ or $\prod (\lambda - 1)$ expresses the integral $\int y^{\lambda-1} e^{-y} dy$ from $y = 0$ to $y = \infty$ or writing λ for $\lambda - 1$, $\prod \lambda$ is the value of the integral $\int y^\lambda e^{-y} dy$ from $y = 0$ to $y = \infty$ if $\lambda + 1$ is a positive quantity.

More generally, putting $y = z^\alpha$ and $\alpha\lambda + \alpha - 1 = \beta$, $\int y^\lambda e^{-y} dy$ becomes $\int \alpha z^\beta e^{-z^\alpha} dz$ which taken between the limits $z = 0$ and $z = \infty$ is therefore expressed by $\prod \left(\dfrac{\beta + 1}{\alpha} - 1 \right)$, or

The value of the integral $\int z^\beta e^{-z^\alpha} dz$ from $z = 0$ and $z = \infty$ becomes $= \dfrac{\prod \left(\frac{\beta+1}{\alpha} - 1 \right)}{\alpha} = \dfrac{\prod \frac{\beta+1}{\alpha}}{\beta + 1}$ if only α and $\beta + 1$ are positive quantities (if either is negative, the integral will be expressed by $-\dfrac{\prod \frac{\beta+1}{\alpha}}{\beta + 1}$). Thus e.g. for $\beta = 0$, $\alpha = 2$, the value of the integral $\int e^{-zz} dz$ is found to be $= \prod \frac{1}{2} = \frac{1}{2}\sqrt{\pi}$.

29.

Euler has obtained for the sum of logarithms $\log 1 + \log 2 + \log 3 +$ etc. $+ \log z$, the series

$$(z + \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \frac{\mathfrak{A}}{1.2z} - \frac{\mathfrak{B}}{3.4z^3} + \frac{\mathfrak{C}}{5.6z^5} - \text{ etc.}$$

where $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$ etc. are Bernoulli's numbers. Therefore $\log \prod z$ is expressed by that series; for even though this conclusion seems at first sight to be restricted to integral values, yet it will be found on considering the matter more closely that the development given by Euler (Instit. Calc. Diff. Cap. VI 159) is certainly just as applicable to positive fractional values as to integers; for he only supposes that the function of z to be developed in series is such that its decrement if z changes into $z - 1$ can be represented by Taylor's theorem at the same time that the same decrement $= \log z$. The former conditioni depends upon the *continuity* of the function so that it does not hold for negative values of z to which therefore that series may not extend; but the latter condition is applicable generally to the function $\log \prod z$ without restriction to the integral values of z. We shall therefore put

$$[58] \quad \log \prod z = (z + \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + \frac{\mathfrak{A}}{1.2z} - \frac{\mathfrak{B}}{3.4z^3} + \frac{\mathfrak{C}}{5.6z^5} - \frac{\mathfrak{D}}{7.8z^7} + \text{ etc.}$$

Since from this we have

$$\log \prod 2z = (2z + \tfrac{1}{2}) \log 2z - 2z + \tfrac{1}{2} \log 2\pi + \frac{\mathfrak{A}}{1.2.2z} - \frac{\mathfrak{B}}{3.4.8z^3} + \frac{\mathfrak{C}}{5.6.32z^5} - \frac{\mathfrak{D}}{7.8.128z^7} + \text{ etc.}$$

and since putting $n = 2$ in formula 57, it becomes

$$\log \prod z - \tfrac{1}{2} = \log \prod 2z - \log \prod z - (2z + \tfrac{1}{2}) \log 2 + \tfrac{1}{2} \log 2\pi$$

we obtain

$$[59] \quad \log \prod z - \tfrac{1}{2} = z \log z - z - \tfrac{1}{2} \log 2\pi - \frac{\mathfrak{A}}{1.2.2z} + \frac{7\mathfrak{B}}{3.4.8z^3} - \frac{31\mathfrak{C}}{5.6.32z^5} + \frac{127\mathfrak{D}}{7.8.128z^7} - \text{ etc.}$$

[3] Hae duae series pro valoribus magnis ipsius z ab initio satis promte conver-

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[3] I have thought it best to leave this paragraph in the original Latin, as I cannot make clear sense of some parts of it: of Gauss Werke Goettingen 1876 Edition, *limitatem* in the fourth line is probably a misprint; the word *hypergeometrical* is also used in a sense somewhat different from its modern use. The general meaning of the paragraph will be clear from the following note by my friend Prof. R. Fujisawa. **D. K.**

We have

$$(1) \quad \log \prod (x+1) = \tfrac{1}{2} \log (2\pi) - x + (x + \tfrac{1}{2}) \log x + \frac{B_1}{1.2} \cdot \frac{1}{x} - \frac{B_2}{3.4} \cdot \frac{1}{x^3} + \cdots$$

$$\cdots + (-1)^{n-1} \frac{B_n}{(2n-1).2n} \cdot \frac{1}{x^{2n-1}} + (-1)^n \theta \frac{B_{n+1}}{(2n+1).(2n+2)} \cdot \frac{1}{x^{2n+1}}$$

$0 < \theta < 1$

Let n increase beyond all limits and we obtain

$$(2) \quad \log \prod (x+1) = \tfrac{1}{2} \log (2\pi) - x + (x + \tfrac{1}{2}) \log x + \frac{B_1}{1.2} \cdot \frac{1}{x} - \frac{B_2}{3.4} \cdot \frac{1}{x^3} + \cdots$$

$$\cdots + (-1)^{n-1} \frac{B_n}{(2n-1).2n} \cdot \frac{1}{x^{2n-1}} + \cdots,$$

where $B_1, B_2, \ldots$ denote Bernoulli's numbers

$$\frac{B_n}{1.2.3.\ldots.2n} = \frac{1}{2^{2n-1}\pi^{2n}} \left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right).$$

The general term in the series **(2)** may be written as the product of the following two expressions

$$\frac{1}{2\pi x} \cdot \frac{2}{2\pi x} \cdot \frac{3}{2\pi x} \cdots \frac{2n-2}{2\pi x}$$

and

$$\frac{1}{2\pi^2 x} \left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right).$$

However great x may be, the first of these factors increases beyond all limits with increasing n, while the second factor converges with increasing n, toward the limits

$$\frac{1}{2\pi^2 \nu x}.$$

gunt, ita ut summam approximatam commode satisque exacte colligere liceat: attamen probe notandum est, pro quovis valore dato ipsius z, quantumvis magno, praecisionem limitatem tantummodo obtineri posse, quum numeri Bernoulliani seriem hypergeometricam constituant, adeoque series illae, si modo satis longe extendantur, certo e convergentibus divergentes evadant. Ceterum negari nequit, theoriam talium serierum divergentium adhuc quibusdam difficultatibus premi, de quibus forsan alia occasione pluribus commentabimur.

30.

From formula 38,

$$\frac{\prod(k, z+\omega)}{\prod(k, z)} = \frac{z+1}{z+1+\omega} \cdot \frac{z+2}{z+2+\omega} \cdot \frac{z+3}{z+3+\omega} \cdots \frac{z+k}{z+k+\omega} k^{\omega}$$

whence taking logarithms and expanding in infinite series, we get

$$[60]\ \log\prod(k, z+\omega) = \log\prod(k, z)$$

$$+\omega\left(\log k - \frac{1}{z+1} - \frac{1}{z+2} - \frac{1}{z+3} - \&\text{ c} - \frac{1}{z+k}\right)$$

$$+\tfrac{1}{2}\omega\omega\left(\frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \frac{1}{(z+3)^2} + \&\text{ c} + \frac{1}{(z+k)^2}\right)$$

$$+\tfrac{1}{3}\omega^3\left(\frac{1}{(z+1)^3} + \frac{1}{(z+2)^3} + \frac{1}{(z+3)^3} + \&\text{ c} + \frac{1}{(z+k)^3}\right)$$

$$+\text{ etc. ad. inf.}$$

The coefficient of ω, which can, if we please, also be put into the form

$$-\frac{1}{z+1} + \log 2 - \frac{1}{z+2} + \log\frac{3}{2} - \frac{1}{z+3} + \log\frac{4}{3} - \frac{1}{z+4} + \log\frac{5}{4}\text{ etc.}$$

$$+\log\frac{k}{k-1} - \frac{1}{z+k}$$

is a series of a finite number of terms; but when k increases indefinitely, it converges to a certain limit, which presents to us a new species of transcendental functions, hereafter to be denoted by Ψz.

It follows that the series **(2)** diverges, however great x may be.

However, notwithstanding that the series **(2)** diverges, it furnishes a ready and exact means of calculating $\log\prod(1+x)$ to any degree of approximation. The approximation is so much the closer as x is large. In fact when x is greater than unity, the terms decrease at first and we see from **(1)** , that the error arising by breaking off the series at any term is less than the first term neglected. Thus we obtain the closest approximation by breaking off the series at the term which just precedes the term of the smallest numerical value.

I take this opportunity of introducing to the reader's notice a German translation of this famous paper of Gauss which has only recently been published in Germany. **R. F.**

Moreover denoteing the sums of the following series continued to infinity

$$\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\frac{1}{(z+3)^2}+\text{ etc.}$$

$$\frac{1}{(z+1)^3}+\frac{1}{(z+2)^3}+\frac{1}{(z+3)^3}+\text{ etc.}$$

$$\frac{1}{(z+1)^4}+\frac{1}{(z+2)^4}+\frac{1}{(z+3)^4}+\text{ etc.}$$

respectively by P, Q, R, etc. (for which it seems not so necessary to introduce functional symbols), we shall have

[61] $\log\prod(z+\omega)=\log\prod z+\omega\Psi z+\frac{1}{2}\omega\omega P-\frac{1}{3}\omega^3 Q+\frac{1}{4}\omega^4 R-\text{ etc.}$

The function Ψz is evidently the 1st derived function of the function $\log\prod z$, so that

[62] $\dfrac{d\prod z}{dz}=\prod z.\Psi z$

Hence $\qquad P=\dfrac{d\Psi(z)}{dz},\quad Q=-\dfrac{dd\Psi z}{2dz^2}\quad R=+\dfrac{d^3\Psi z}{2dz^3}$, etc.

31.

The function Ψz is almost as remarkable as the function $\prod z$; we shall therefore collect here some of its more important relations. By differentiation, the equation 44 becomes

[63] $\Psi(z+1)=\Psi z+\dfrac{1}{z+1}$

whence we have

[64] $\Psi(z+n)=\Psi z+\dfrac{1}{z+1}+\dfrac{1}{z+2}+\dfrac{1}{z+3}+\text{ etc. }+\dfrac{1}{z+n}$

By means of this formula, we may proceed forward from less to greater values of z, or backward from greater to less values. For greater positive values of z, the numerical values of the function can be calculated conveniently enough by the following formulae obtained by the differentiation of the equations 58, 59, to which, however, the same remarks apply as those made in Art. 29 about the formula 58 and 59.

[65] $\Psi z = \log z + \frac{1}{2z} - \frac{\mathfrak{A}}{2zz} + \frac{\mathfrak{B}}{4z^4} - \frac{\mathfrak{C}}{6z^6} + \text{etc.}$

[66] $\Psi(z - \frac{1}{2}) = \log z + \frac{\mathfrak{A}}{2.2zz} - \frac{\mathfrak{B}}{4.8z^4} + \frac{\mathfrak{C}}{6.32z^6} + \text{etc.}$[4]

Thus for $z = 10$, we have calculated

$$\Psi z = 2.35175258906672110764743$$

whence we proceed backwards to

$$\Psi 0 = -0.57721566490153286060653$$[5]

For positive integral values of z we have generally

[67] $\Psi z = \Psi 0 + 1 + \frac{1}{2} + \frac{1}{3} + \text{etc. } + \frac{1}{z}$

For negative integral values on the other hand, Ψz becomes clearly an infinitely large quantity.

32.

The formula 55 gives us $\log \prod(-z) + \log \prod(z-1) = \log \pi - \log \sin z\pi$ whence by differention, we get

[68] $\Psi(-z) - \Psi(z-1) = \pi \cot z\pi$

And since from the definition of the function Ψ, we have generally

[69] $\Psi(x) - \Psi(y) = -\frac{1}{x+1} + \frac{1}{y+1} - \frac{1}{x+2} + \frac{1}{y+2} - \frac{1}{x+3} + \text{etc.},$

this gives rise to the well known series

$$\pi \cot z\pi = \frac{1}{z} - \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{2-z} + \frac{1}{2+z} - \frac{1}{3-z} + \text{etc.},$$

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[5] Since this value does not agree beyond the 20th figure from that calculated by Mascheroni in Notes to Euler's Integral Calculus, I persuaded F. B. G. Nicolai, a young man indefatigable in calculation to repeat that computation and to extend it further. Thus he found by double calculation, namely descending from $z = 50$ and from $z = 100$,

$$\Psi 0 = -0.5772156649015328606065120900824024510421$$

To the same expert calculator also is due the 2nd part of the tables annexed to the end of this section, giving the values of the function Ψz to 18 figures (of which the last is not certain) for every 100th value of z from 0 to 1. But the methods by which both tables have been constructed depend partly upon the theorems given here, pertly on remarkable artifices of calculation, which we shall bring forward on another occasion. [ORIGINAL FOOT-NOTE.]

Similarly the differentiation of formula 57 produces

$$[70]\ \ \Psi z+\Psi\left(z-\frac{1}{n}\right)+\Psi\left(z-\frac{2}{n}\right)+\text{ etc. }+\Psi\left(z-\frac{n-1}{n}\right)=n\Psi nz-n\log n$$

so that putting $z=0$

$$[71]\ \ \Psi\left(-\frac{1}{n}\right)+\Psi\left(-\frac{2}{n}\right)+\left(-\frac{3}{n}\right)+\text{ etc. }+\Psi\left(-\frac{n-1}{n}\right)=(n-1)\Psi 0-n\log n$$

Thus, for example, we have

$$\Psi(-\frac{1}{2})=\Psi 0-2\log 2=-1.9635100260214234794409 9,$$

whence again

$$\Psi\frac{1}{2}=+0.0364899739785765205590 1.$$

33.

As in the preceding Art., we have reduced $\Psi(-\frac{1}{2})$ to $\Psi 0$ and a logarithm, so we shall reduce generally $\Psi\left(-\frac{m}{n}\right)$ where m, and n are integers of which m is the less, to $\Psi 0$ and logarithms. Let us put $\frac{2\pi}{n}=\omega$, and let ϕ be equal to any one of the angles $\omega, 2\omega, 3\omega \ldots (n-1)\omega$; whence $1=\cos n\phi=\cos 2n\phi=\cos 3n\phi$ etc., $\cos\phi=\cos(n+1)\phi=\cos(n+2)\phi$ etc.[6] , $\cos 2\phi=\cos(n+2)\phi$ etc., and moreover $\cos\phi+\cos 2\phi+\cos 3\phi+$ etc. $\cos(n-1)\phi+1=0$. Thus we have

$$\cos\phi.\Psi\frac{1-n}{n}=-n\cos\phi+\cos\phi.\log 2-\frac{n}{n+1}\cos(n+1)\phi+\cos\phi.\log\frac{3}{2}-\text{ etc.}$$

$$\cos 2\phi.\Psi\frac{2-n}{n}=-\frac{n}{2}\cos 2\phi+\cos 2\phi.\log 2-\frac{n}{n+2}\cos(n+2)\phi+\cos 2\phi.\log\frac{3}{2}-\text{ etc.}$$

$$\cos 3\phi.\Psi\frac{3-n}{n}=-\frac{n}{3}\cos 3\phi+\cos 3\phi.\log 2-\frac{n}{n+3}\cos(n+3)\phi+\cos 3\phi.\log\frac{3}{2}-\text{ etc.}$$

etc. up to

$$\cos(n-1)\phi.\Psi\left(-\frac{1}{n}\right)=-\frac{n}{n-1}\cos(n-1)\phi+\cos(n-1)\phi.\log 2-\frac{n}{2n-1}\cos(2n-1)\phi+\cos(n-1)\phi.\log\frac{3}{2}-\text{ et}$$

$$\Psi 0=-\frac{n}{n}\cos n\phi+\log 2-\frac{n}{2n}\cos 2n\phi+\log\frac{3}{2}-\text{ etc.}$$

and by *summation*,

$$\cos\phi.\Psi\frac{1-n}{n}+\cos 2\phi.\Psi\frac{2-n}{n}+\cos 3\phi.\Psi\frac{3-n}{n}+\text{ etc. }+\cos(n-1)\phi.\Psi\left(-\frac{1}{n}\right)+\Psi 0$$

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[6] $(n+2)\phi$ should evidently be $(2n+1)\phi$. **D. K.**

$$= -n(\cos\phi + \frac{1}{2}\cos 2\phi + \frac{1}{3}\cos 3\phi + \frac{1}{4}\cos 4\phi + \text{ etc. } \textit{ad inf.})$$

But we have generally for the value of x not greater than unity,

$$\log(1 - 2x\cos\phi + xx) = -2(x\cos\phi + \frac{1}{2}xx\cos 2\phi + \frac{1}{3}x^3\cos 3\phi + \text{ etc.})$$

which series follows easily from the development of $\log(1 - rx) + \log\left(1 - \frac{x}{r}\right)$, where r stands for the quantity $\cos\phi + \sqrt{-1}.\sin\phi$. Hence the preceding equation becomes

$$[72]\ \cos\phi.\Psi\frac{1-n}{n} + \cos 2\phi.\Psi\frac{2-n}{n} + \cos 3\phi.\Psi\frac{3-n}{n} + \text{ etc.}$$

$$+ \cos(n-1)\phi.\Psi\left(-\frac{1}{n}\right) = -\Psi 0 + \tfrac{1}{2}n\log(2 - 2\cos\phi)$$

In this equation, put successively $\phi = \omega$, $\phi = 2\omega$, $\phi = 3\omega$, etc. up to $\phi = (n-1)\omega$; and multiply the equations so obtained respectively by $\cos m\omega$, $\cos 2m\omega$, $\cos 3m\omega$, etc., up to $\cos(n-1)m\omega$; and add the sum of the products so formed to equation 71

$$\Psi\frac{1-n}{n} + \Psi\frac{2-n}{n} + \Psi\frac{3-n}{n} + \text{ etc.} + \Psi\left(-\frac{1}{n}\right) = (n-1)\Psi 0 - n\log n.$$

Remembering that

$$1 + \cos m\omega.\cos k\omega + \cos 2m\omega.\cos 2k\omega + \cos 3m\omega.\cos 3k\omega$$

$$+ \text{ etc. } \cos(n-1)m\omega.\cos(n-1)k\omega = 0$$

where k denotes any one of the numbers $1, 2, 3 \ldots (n-1)$ except the two, m and $n-m$, for which the sum becomes $= \frac{1}{2}n$, it is evident that the result of the summation of those equations becomes after division by $\frac{n}{2}$

$$[73]\ \Psi\left(-\frac{m}{n}\right) + \Psi\left(-\frac{n-m}{n}\right) =$$

$$2\Psi 0 - 2\log n + \cos m\omega.\log(2 - 2\cos\omega) + \cos 2m\omega.\log(2 - 2\cos 2\omega)$$

$$+ \cos 3m\omega.\log(2 - 2\cos 3\omega) + \text{ etc. } + \cos(n-1)m\omega.\log(2 - 2\cos(n-1)\omega)$$

Clearly the last term of this equation $= cosm\omega.\log(2 - 2\cos omega)$, the last but one $= cos2m\omega.\log(2 - 2\cos 2omega)$ etc., so that the terms are always equal two by two, except, if n be even, the single term $\cos\frac{n}{2}m\omega.\log\left(2 - 2\cos\frac{n}{2}\omega\right)$, which $= +2\log 2$ for m even, or $= -2\log 2$ for m odd. Now combining with equation 73 the following equation

$$\Psi\left(-\frac{m}{n}\right) + \Psi\left(-\frac{n-m}{n}\right) = \pi\cot\frac{m}{n}\pi$$

we have, for the odd values of n, since m is a positive integer less than n,

$$[74]\ \Psi\left(-\frac{m}{n}\right) = \Psi 0 + \tfrac{1}{2}\pi\cot\frac{m\pi}{n} - \log n + \cos\frac{2m\pi}{n}.\log\left(2 - 2\cos\frac{2\pi}{n}\right)$$

$$+\cos\frac{4m\pi}{n}.\log\left(2 - 2\cos\frac{4\pi}{n}\right) + \cos\frac{6m\pi}{n}.\log\left(2 - 2\cos\frac{6\pi}{n}\right) + \text{etc.}$$

$$+\cos\frac{(n-1)m\pi}{n}.\log\left(2 - 2\cos\frac{(n-1)\pi}{n}\right)$$

And for the even values of n,

$$[75]\ \Psi\left(-\frac{m}{n}\right) = \Psi 0 + \tfrac{1}{2}\pi\cot\frac{m\pi}{n} - \log n + \cos\frac{2m\pi}{n}.\log\left(2 - 2\cos\frac{2\pi}{n}\right)$$

$$+\cos\frac{4m\pi}{n}.\log\left(2 - 2\cos\frac{4\pi}{n}\right) + \text{etc. } + \cos\frac{(n-2)m\pi}{n}.\log\left(2 - 2\cos\frac{(n-2)\pi}{n}\right)$$

$$\pm\log 2$$

where the upper sign is to be taken for m even, and the lower for m odd. Thus, for example, we obtain

$$\Psi(-\tfrac{1}{4}) = \Psi 0 + \tfrac{1}{2}\pi - 3\log 2, \qquad \Psi(-\tfrac{3}{4}) = \Psi 0 - \tfrac{1}{2}\pi - 3\log 2$$

$$\Psi(-\tfrac{1}{3}) = \Psi 0 + \tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}} - \tfrac{3}{2}\log 3, \quad \Psi(-\tfrac{2}{3}) = \Psi 0 - \tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}} - \tfrac{3}{2}\log 3.$$

And further, these equations being combined with equation 64, it is at once evident that Ψz can generally be determined for *any rational value whatsoever* of z, positive or negative, by $\Psi 0$ and the logarithms, certainly a most noteworthy theorem.

34.

Since, by Art. 28, $\prod\lambda$ is the value of the integral $\int y^\lambda e^{-y}dy$, from $y = 0$ to $y = \infty$, if $\lambda + 1$ be a positive quantity, by differentiating with respect to λ, we have

$$\frac{d\prod\lambda}{d\lambda} = \frac{d\int y^\lambda e^{-y}dy}{d\lambda} = \int y^\lambda e^{-y}\log y dy$$

or

[76] $\prod \lambda.\Psi\lambda = \int y^\lambda e^{-y} \log y.dy$, from $y = 0$ to $y = \infty$.

Generally putting $y = z^\alpha$, $\alpha\lambda + \alpha - 1 = \beta$, the value of the integral $intz^\beta e^{-z^\alpha} \log z dz$, from $z = 0$ to $z = \infty$, becomes

$$= \frac{1}{aa} \prod \left(\frac{\beta+1}{\alpha} - 1\right).\Psi\left(\frac{\beta+1}{\alpha} - 1\right) = \frac{1}{\alpha(\beta+1)} \prod \frac{\beta+1}{\alpha}.\Psi\frac{\beta+1}{\alpha} - \frac{1}{(\beta+1)^2} \prod \frac{\beta+1}{\alpha}$$

if $\beta + 1$ and α be both positive quantity, or if either $\beta + 1$ or α be negative, equal to the same quantity with opposite sign.

35.

But not only the product $\prod \lambda.\Psi\lambda$, but also the function $\Psi\lambda$ itself can be expressed as definite integral. k being a positive integer, it is evident that the value of the integral $\int \frac{x^\lambda - x^{\lambda+k}}{1-x} dx$, from $x = 0$ to $x = 1$, is

$$= \frac{1}{\lambda+1} + \frac{1}{\lambda+2} + \frac{1}{\lambda+3} + \text{ etc. } + \frac{1}{\lambda+k}$$

Further, since the value of the indefinite integral $\int \left(\frac{1}{1-x} - \frac{kx^{k-1}}{1-x^k}\right) dx =$ Const. $+\log \frac{1-x^k}{1-x}$, the same between the limits $x = 0$ and $x = 1$ will be $\log k$, whence evidently the value of the integral $S = \int \left(\frac{1 - x^\lambda + x^{\lambda+k}}{1-x} - \frac{kx^{k-1}}{1-x^k}\right) dx$ between the same limits is

$$= \log k - \frac{1}{\lambda+1} - \frac{1}{\lambda+2} - \frac{1}{\lambda+3} - \text{ etc. } - \frac{1}{\lambda+k}$$

which expression we shall denote by Ω. Let us separate S into two parts

$$\int \left(\frac{1-x^\lambda}{1-x}\right) dx + \int \left(\frac{x^{\lambda+k}}{1-x} - \frac{kx^{k-1}}{1-x^k}\right) dx$$

The first part $\int \frac{1-x^\lambda}{1-x} dx$, transforms by putting $x = y^k$ into

$$\int \frac{ky^{k-1} - ky^{\lambda k+k-1}}{1-y^k} dy$$

whence evidently the value of the above integral from $x = 0$ to $x = 1$ is equal to the value of the integral

$$\int \frac{kx^{k-1} - kx^{\lambda k+k-1}}{1-x^k} dx$$

between the same limits, since the letter y may clearly under this restriction be changed into x. Hence, the integral S becomes, between the same limits,

$$\int \left(\frac{x^{\lambda+k}}{1-x} - \frac{kx^{\lambda k+k-1}}{1-x^k} \right) dx$$

But putting $x^k = z$, this integral becomes

$$\int \left(\frac{z^{\frac{\lambda+1}{k}}}{k(1-z^{\frac{1}{k}})} - \frac{z^\lambda}{1-z} \right) dz$$

which therefore taken between the limits $z = 0$ and $z = 1$ is equal to Ω. But when k increases indefinitely, the limit of Ω is $\Psi\lambda$, the limit of $\frac{\lambda+1}{k}$ is 0, and the limit of $k(1-z^{\frac{1}{k}})$ is $\log \frac{1}{z}$ or $-\log z$. Wherefore, we have

[77] $\Psi\lambda = \int \left(\frac{1}{\log \frac{1}{z}} - \frac{z^\lambda}{1-z} \right) dz = \int \left(-\frac{1}{\log z} - \frac{z^\lambda}{1-z} \right) dz$

from $z = 0$ to $z = 1$.

36.

Definite integrals, by which the functions $\prod \lambda, \prod \lambda \cdot \Psi\lambda$ have been expressed above, must be restricted to such values of λ that $\lambda + 1$ is a positive quantity. This restriction arises from their very derivation, and indeed it can be easily seen that for other values of λ those integrals will always become infinite, even though the functions $\prod \lambda, \prod \lambda \cdot \Psi\lambda$ may remain finite. The truth of formula 77 ought certainly to be subject to the same condition that $\lambda + 1$ should be positive (for otherwise the integral becomes infinite even though the function $\Psi\lambda$ remains finite): but the derivation of the formula seem at first sight to be subject to no resriction. But if we examine more closely, it can be easily seen that this restriction exists in the very analysis by which the formula has been obtained. For we have tacitly assumed the integral $\int \frac{1-x^\lambda}{1-x} dx$ for which we have substituted its equivelent $\int \frac{kx^{k-1} - kx^{\lambda k+k-1}}{1-x^k} dx$, to have a *finite* value, which condition requires that $\lambda + 1$ should be a positive quantity. Indeed it follows from our analysis that those two integrals are always equal if the latter extend from $x = 0$ to $x = 1 - \omega$, and the former from $x = 0$ to $x = (1-\omega)^k$, however small quantity ω may be provided only it is not $= 0$: but notwithstanding this in the case when $\lambda + 1$ is not a positive quantity, the two integrals extended from $x = 0$ to the *same* limit $x = 1 - \omega$ by no means tend to equality but rather their difference then increases infinitely when ω diminishes indefinitely. This example shows how much circumspection is necessary in dealing with infinite quantities which are in our opinion to be admitted to analytical reasoning only so far as they can be reduced to the theory of limits.

37.

Putting in formula 77, $z = e^{-u}$, it can clearly be also put in the form

$$\Psi\lambda = -\textstyle\int \left(\frac{e^{-u}}{u} - \frac{e^{-u\lambda - u}}{1 - e^{-u}} \right) du \text{ from } u = \infty \text{ to } u = 0, \text{ that is,}$$

$$[78]\ \ \Psi\lambda = -\textstyle\int \left(\frac{e^{-u}}{u} - \frac{e^{-\lambda u}}{e^{u} - 1} \right) du \text{ from } u = 0 \text{ to } u = \infty.$$

(Thence the value of $\prod \lambda$ obtained in Art. 28 can be transformed by putting $e^{-y} = v$, into the following,

$$\prod \lambda = \int \left(log \frac{1}{v} \right)^{\lambda} dv, \text{ from } v = 0 \text{ to } v = 1.$$

Further, from formula 77, we have at once,

$$[79]\ \ \Psi\lambda - \Psi\mu = \textstyle\int \frac{z^{\mu} - z^{\lambda}}{1 - z} dz \text{ from } z = 0 \text{ to } z = 1$$

where besides $\lambda + 1$, $\mu + 1$ also ought to be a positive quantity.

Putting in the same formula 77, $z = u^{\alpha}$ where α is a positive quantity, we have

$$\Psi\lambda = \int \left(-\frac{u^{\alpha - 1}}{\log u} - \frac{\alpha u^{\alpha\lambda + \alpha - 1}}{1 - u^{\alpha}} \right) du \text{ from } u = 0 \text{ to } u = 1$$

and since we may put, for positive value of β,

$$\Psi\lambda = \int \left(-\frac{u^{\beta - 1}}{\log u} - \frac{\beta u^{\beta\lambda + \beta - 1}}{1 - u^{\beta}} \right) du$$

we get,

$$0 = \int \left(\frac{u^{\alpha - 1} - u^{\beta - 1}}{\log u} + \frac{\alpha u^{\alpha\lambda + \alpha - 1}}{1 - u^{\alpha}} - \frac{\beta u^{\beta\lambda + \beta - 1}}{1 - u^{\beta}} \right) du$$

or

$$\int \frac{u^{\alpha - 1} - u^{\beta - 1}}{\log u} du = \int \left(\frac{\beta u^{\beta\lambda + \beta - 1}}{1 - u^{\beta}} - \frac{\alpha u^{\alpha\lambda + \alpha - 1}}{1 - u^{\alpha}} \right) du$$

the integration extending in all cases from $u = 0$ to $u = 1$. But putting $\lambda = 0$, the latter can be integrated *indefinitely*; namely it $= \log \frac{1 - u^{\alpha}}{1 - u^{\beta}}$, if it is to vanish for $u = 0$; therefore since for $u = 1$, we ought to put $\frac{1 - u^{\alpha}}{1 - u^{\beta}} = \frac{\alpha}{\beta}$, we get $\log \frac{\alpha}{\beta} = \int \frac{u^{\alpha - 1} - u^{\beta - 1}}{\log u} du$, from $u = 0$ to $u = 1$, which theorem has been already obtained by other methods by the illustrious Euler.
