Goettingen gelehrte Anzeigen. 1812 February 10.[1]

The logarithmic and circular functions, being the simplest types of transcendental functions, are those which the analysts have most concerned themselves with. They earn this honor both because of their constant presence in almost every mathematical investigation, theoretical and practical, and [also] because of the almost inexhaustible wealth of interesting truths which their theory offers. Other transcendental functions, which have been far less cultivated, can not be reduced to these, but must rather be considered as their own, higher species, to which the former are related only as a special case. And yet, many such functions are not less fruitful in interesting relations, and hence, for those who honor Analysis in itself, are not less important; also, their frequent occurrence in sundry other investigations must commend them to those who like to first ask about their practical uses.[2] Professor Gauss has already occupied himself for many years with investigations of those same higher transcendental functions, for which far extended results have been confirmed. He had delivered a part of this (only a very small portion, to be sure) as a lecture to the Royal Society of Science on January 30, under the title

Disquisitiones generales circa seriem infinitam

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \alpha + \beta \cdot \beta + 1}{1 \cdot 2 \cdot \gamma \cdot \gamma + 1} xx + \frac{\alpha \cdot \alpha + 1 \cdot \alpha + 2 \cdot \beta \cdot \beta + 1 \cdot \beta + 2}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot \gamma + 1 \cdot \gamma + 2} x^3 + \text{ etc.}$$

Pars Prior,

which can be seen almost as a preamble to a series of essays to be given in the future. We shall report the key points of content here.

Of course, the transcendental functions have their true source, either obvious or hidden, in the infinite. The operations of integration, of the summation of infinite series, of the development of infinite products, of carrying out infinitely

[1] 1

[2] Und doch sind manche solcher Functionen nicht minder fruchtbar an interessanten Relationen, und daher dem, welcher die Analyse um ihrer selbst willen ehrt, nicht minder wichtig; so wie ihr hufiges Vorkommen bei mancherlei andern Untersuchungen sie dem empfehlen muss, der gern erst nach pratischem Nutzen fragt.

continued fractions, or, in general, the approximation of a boundary through operations set forth without end according to determined laws these are the proper foundations from which the transcendental functions will arise, or, if one will rather be served another image, these are the proper paths upon which one arrives at them. Sundry such paths commonly lead to one end: the circumstances and the object that are set before it, must determine that [path] which will first or by preference be chosen. The series that forms the subject of the present essay is of a very extensive generality. Depending on how the magnitudes α , β , γ (which, next to x, the author distinguishes by the designation first, second, third, fourth *element*, thus he denotes the function, represented by the whole series, with the symbol $F(\alpha,\ \beta,\ \gamma\ ,\ x)$) are determined, the series represents algebraic, logarithmic, trigonometric, or higher transcendental functions, and it can be asserted that, up to now, there were scarcely any transcendental function investigated by analysts that could not be reduced to this series. A great number of truths, which indeed have been found related to those functions already drawn into consideration, can be derived from the general nature of the functions represented through our series, and if only because of this, their investigation would certainly merit the attention of the geometer, though the beginnings of the approach to new truths is to be considered as the prime objective.[3] The present essay contains only the first half of the general investigations of the author, which makes up a necessary part of the great bulk. Here, we take precisely this series itself as the origin of the transcendental functions, which, in further course of the work, will be derived from a more general practicable source, and will be considered from a higher point of view. The former approach [*Erzeugung*] makes necessary, according to its nature, the limitation to the cases where the series converges, where thus the fourth element x, positive or negative, does not exceed the value 1: the latter approach [*Erzeugungsart*] will remove this limitation. However, even the former approach leads already to a multitude of remarkable truths of a convenient and almost more elementary path, and, because of that, the author made this the beginning.

The first half of the investigation is divided into three sections, in which several general remarks are set out at the beginning. In order to provide a sample of the extended application of the series, to begin with, 23 distinct series expansions of algebraic, logarithmic and trigonometric functions are reduced to the same. For an example of higher transcendental functions, the coefficients arising from the expansion of $(aa + bb - 2ab\cos\phi)^n$, which progress according to the multiples of ϕ are given in three different ways.[4]

The *first section* concerns itself with the relations between such series of the above form, in which the value of one of the three first elements varies by one

[3]Eine grosse Menge von Wahrheiten, welche in Beziehung auf solche schon in Betractung gezogene Functionen schon aufgefunden sind, lassen sich aus der allgemeinen Natur der durch unsere Reihen dargestellten Function ableiten, und schon um desswillen wuerden Untersuchungen darueber die Aufmerksamkeit der Geometer verdienen, obwohl diess nur als Neben sache, und die Eroeffnung des Zuganges zu neuen Wahrheiten als Hauptzweck zu betrachten ist.

[4]In the paper referred to, Gauss investigates the function to the -n power. The function cited here thus contains a typo. (pjm)

unit, while the values of the three others remain the same. The author calls this series *series contiguae*, which in German could be called *verwandte Reihen* [relative series]. Thus, to this series $F(\alpha, \beta, \gamma, x)$ stand six related [ones], namely $F(\alpha+1, \beta, \gamma, x)$, $F(\alpha-1, \beta, \gamma, x)$, $F(\alpha, \beta+1, \gamma, x)$, $F(\alpha, \beta-1, \gamma, x)$,$F(\alpha, \beta, \gamma+1, x)$, $F(\alpha, \beta, \gamma-1, x)$, and it will be shown here that there exists a linear equation between the first and every two of the relatives. Fifteen equations arise in this way. The important theorem follows from this, that, if $\alpha' - \alpha$, $\alpha'' - \alpha$, $\beta' - \beta$, $\beta'' - \beta$, $\gamma' - \gamma$, $\gamma'' - \gamma$, are whole numbers, a linear equation also occurs between $F(\alpha, \beta, \gamma, x)$, $F(\alpha', \beta', \gamma', x')$, $F(\alpha'', \beta'', \gamma'', x'')$, and thus, generally speaking, from the values of two of these functions, the value of the third can be derived. The author has compiled here some of the easiest or most noteworthy cases in particular.

The *second section* gives the transformation into continuous fractions, and indeed the quotients

$$\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}, \ \frac{F(\alpha, \beta+1, \gamma, x)}{F(\alpha, \beta, \gamma, x)}, \ \frac{F(\alpha-1, \beta+1, \gamma, x)}{F(\alpha, \beta, \gamma, x)}$$

can be reduced to three others through the obviously allowed substitution of the two first elements. Almost every expansion into continuous fractions hitherto known is but a special case of this theorem. Especially noteworthy is the case where β in the second expansion is set $= 0$. A theorem follows from this, which we add here on account of its broad application. The function $F(\alpha, 1, \gamma, x)$ or, what is the same, the series

$$1 + \frac{\alpha}{\gamma}x + \frac{\alpha \cdot \alpha+1}{\gamma \cdot \gamma+1}xx + \frac{\alpha \cdot \alpha+1 \cdot \alpha+2}{\gamma \cdot \gamma+1 \cdot \gamma+2}x^3 + \text{etc.}$$

gives the continuous fraction

$$\cfrac{1}{1 - \cfrac{ax}{1 - \cfrac{bx}{1 - \cfrac{cx}{1 - \cfrac{dx}{1 - \text{etc.}}}}}}$$

where the coefficients a, b, c, d etc. proceed according to the following laws

$$\begin{array}{ll} a = \frac{\alpha}{\gamma} & b = \frac{\gamma-\alpha}{\gamma(\gamma+1)} \\ c = \frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)} & d = \frac{2(\gamma+1-\alpha}{(\gamma+2)(\gamma+3)} \\ e = \frac{(\alpha+2)(\gamma+1)}{(\gamma+3)(\gamma+4)} & f = \frac{3(\gamma-2-\alpha)}{(\gamma+4)(\gamma+5)} \text{ etc.} \end{array}$$

According to this, e.g. for the powers of a binomial, or the series for $\log(1+x)$,$\log \frac{1+x}{1-x}$, exponentials, the arctangent or the arcsine, etc., can be transformed into continuous fractions. Also, the transformation given in the Theoria motus corpus coelestium is based on such fractions, whose proof is reproduced by the author.

By far, the largest part of the essay is the third section, in which the series is investigated when the value of the fourth element is set = 1. After first proving with geometrical precision that the series for $x = 1$ converges to a finite sum only when $\gamma - \alpha - \beta$ is a positive magnitude, the author reduces this sum, $F(\alpha, \beta, \gamma, 1)$, to the expression $\frac{\Pi(\gamma-1)\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\Pi(\gamma-\beta-1)}$ where the characterisic Π. indicates a peculiar [eigene] type of transcendental function, whose generation the author bases on an infinite product. This function, of the highest importance in the entirety of Analysis, is fundamentally none other than *Euler's* inexplicable function $\Pi(z) = 1 \cdot 2 \cdot 3 \cdot 4$, but *this* type of approach or definition is, according to the author's opinion, quite unsuitable, since it only has a clear sense for positive, whole number values of z. The foundation chosen by the author is applicable in general, and provides as clear a sense with imaginary values of z as with real, and with them one runs no risk of getting such paradoxes and inconsistencies as the honorable Mr. *Kramp* does with his numerical faculties, which, as can be easily shown, can be reduced to the above function. However, the admission of [*Kramp's* numerical faculties] in Analysis appears less suitable than the admission of the above function, since the former depends on three magnitudes, while the latter depends on only one, and yet must be considered just as general. The author wishes to see these transcendental functions $\Pi(z)$ given their civil rights in analysis, for which purpose, perhaps the choice of a name of their own would be most conducive: the right of this [naming] shall remain for he who makes the most important discoveries in the theory of these functions, which are to be very highly valued for the efforts of the geometer. Here the author has already collected a significant number of remarkable theorems, a part of which is to be considered as new. The space does not allow us to go into much detail about this here: we only select one such theorem, that the value of the integral $\int x^{\lambda-1}(1 - x^{\mu})^{\nu} dx$ from $x = 0$ to $x = 1$ can be easily reduced to the function Π, and that all of the relations for the integral, some of which were painstakingly found by *Euler*, can be derived with greater ease from the general properties of these functions, so inversely $\Pi(z)$ can be represented if z is a rational number through some such determined integral.

Not less remarkable is the function generated from the differentiation of $\Pi(z)$, likewise transcendental, or rather

$$\frac{d \log \Pi(z)}{dz} = \frac{d\Pi(z)}{\Pi(z) \cdot dz}$$

which the author has denoted with $\Psi(z)$, and likewise merits a special appellation [*Benennung*]. On the numerous noteworthy characteristics of this function which are laid out in the essay, we here assert only one, that generally $\Psi(z)$, if z is a rational magnitude, can be reduced to logarithms and circular functions; $\Psi(0)$ itself however, is the known number 0,5772156649K, investigated by Euler and others, taken as negative. The author communicates this number, which he himself calculated out to 23 decimal places, which deviates somewhat from

Mascheronis determination. Moreover $\Pi(z)$ as well as $\Psi(z)$ are associated with more remarkable integrals for determined values of the variable.

This third section is accompanied by a table for $\log \Pi(z)$ and for $\Psi(z)$, calculated with greatest accuracy by Mr. *Nicolai* under the oversight of Professor *Gauss*, wherein the argument z progresses through each and every hundredth from 0 to 1; it is clear that, from the theory of these functions, all other values of z can easily be reduced to these.