Gauss to Encke

Göttingen

December 24, 1849

Translated from the German by Peter Martinson

Most Honored Friend!

...The kind communication of your remarks on the frequency of prime numbers was interesting to me in more than one respect. You have reminded me of my own pursuit of the same subect, whose first beginnings occurred a very long time ago, in 1792 or 1793, when I had procured for myself Lambert's supplement to the table of logarithms. Before I had occupied myself with the finer investigations of higher arithmetic, one of my first projects was to direct my attention to the decreasing frequency of prime numbers, to which end I counted them up in several chiliads and recorded the results on one of the affixed white sheets. I soon recognized, that under all variations of this frequency, on average, it is nearly inversely proportional to the logarithm, so that the number of all prime numbers under a given boundary n were nearly expressed through the integral

$$\int \frac{dn}{\log n},$$

if the logarithm is understood to be hyperbolic. At a later time, when the list to 400031, printed in Vega's table (of 1796), was made known to me, I extended my count further, which verified that relationship. The 1811 appearance of Chernac's cribrum [seive] gave me great joy, and (since I did not have the patience for a continuous count of the series) I have very often employed a spare unoccupied quarter of an hour in order to count up a chiliad here and there; however, I eventually dropped it completely, without having quite completed the first million. Some time later I employed Goldschmidt's dilligence, partially to fill the gaps still remaining in the first million, partially to continue the enumeration farther beyond Burckhardt's table. So have the first three million been counted (now already for many years), and compared with the integral value. I lay out here only a small extract:

Under	Number of Prime Numbers	Integral $\int \frac{dn}{\log\ n}$	Dev.		Your Formula	Dev.	
500000	41556	41606.4	+	50.4	41596.9	+	40.9
1000000	78501	79627.5	+	126.5	78672.7	+	171.7
1500000	114112	114263.1	+	151.1	114374.0	+	262.0
2000000	148883	149054.8	+	171.8	149233.0	+	350.0
2500000	183016	183245.0	+	229.0	183495.1	+	479.1
3000000	216745	216970.6	+	225.6	217308.5	+	563.5

I was not aware that LEGENDRE had concerned himself with the same subject; on occasion of your letter, I have looked into his *Théorie des Nombres*, and in the second edition found a single page on this, which I must have overlooked before (or since forgotten). LEGENDRE used the formula

$$\frac{n}{\log\ n - A}$$

where A shall be a constant that he sets equal to 1.08366. From a cursory calculation for the above cases, I find the deviation

$$\begin{array}{ll} - & 23.3 \\ + & 42.2 \\ + & 68.1 \\ + & 92.8 \\ + & 159.1 \\ + & 167.6 \end{array}$$

These differences are smaller than those with the integral, though they do appear to grow more quickly than [the differences given by the integral] with increasing n, so that it is possible that they could easily surpass the latter, if carried out much farther. In order to bring calculation and formula into agreement, instead of 1.08366, A must be set equal to

1.09040
1.07682
1.07582
1.07529
1.07179
1.07297

It appears that the (average) value of A decreases with growing n; but, whether the limit as n grows to infinity will be 1 or a magnitude different than 1, on this I will venture no guess. I cannot say that there is license to expect such a simple boundary value; on the other side, the excess of A over 1 could quite reasonably be a magnitude of the order $\frac{1}{\log\ n}$. I would be inclined to believe that the differential of the function in question, must be easier than the function itself. While I assumed $\frac{dn}{\log\ n}$ for that, LEGENDRE'S formula would be assumed to be a differential function, which would be nearly $\frac{dn}{\log\ n-(A-1)}$. Moreover, your formula would, for a very large n, be considered as agreeing with

$$\frac{n}{\log\ n - \frac{1}{2k}},$$

where k is the modulus of the BRIGGS logarithm, [and] consequently with LEGENDRE'S formula, if

$$A = \frac{1}{2k} = 1.1513.$$

Finally, I shall note that I have found a few differences between your enumeration and mine.

Between	59000	and	60000	You have	95	I have	94
	101000		102000		94		93

The first difference perhaps has its basis in that the prime number 59023 is entered twice in LAMBERT'S supplement. In LAMBERT'S supplement, the chiliad of 101000 – 102000 teems with errors; in my copy, I have marked 7 numbers that are not prime, and against 2 erring I have intercalated. Could you not cause the young DASE to count up the prime numbers in the following million from that table which exists at the academy, which [enumeration] I fear the public be lacking? For this case I note, that in the second and third million, the count be made according to my prescription of a particular scheme, which I myself had also made use of already with a part of the first million. The count of each 100000 is on a (small) octavo page in 10 columns, each gathered into a myriad; in addition, a column comes before (left) and one after (right); as an example, here is a vertical column and the two additional columns for the interval 1000000 ... 1100000

...

The first vertical series, for example, will serve for illustration. In the myriad 1000000 to 1010000 is 100 hecatontads; only one of them contains but 1 prime number; absolutely none with 2 or 3; two parts each containing 4 prime numbers; eleven parts each with 5, etc., all together giving $752 = 1{\cdot}1 + 4{\cdot}2 + 5{\cdot}11 + 6{\cdot}14 + \ldots$. The numbers 14, 15, 16 are in the first vertical series only as overflow, since no hecatontad appears with that many prime numbers; however, on the following sheet they get quantity. Last, the 10 pages will again be combined into 1, and thus comprise the whole second million.

Anyway, it is time to conclude. ...With affectionate wishes for your good health

Constantly Yours

C. F. Gauss