Determination of the Attraction which a Planet would exert upon a Point at an arbitrarily given Location, if its Mass were distributed continuously along the entire Orbit, in proportion to the Time it takes to traverse its individual Parts

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1.

The secular variations which the elements of a planetary orbit undergo owing to the perturbation of another planet, are independent of the position of the latter in its orbit, and their values are the same whether the perturbing planet follows the elliptical path according to the Keplerian laws or whether its mass is considered to be continuously distributed along its orbit such that the sections of the orbit which are traversed in equal times are also given equal amounts of mass, provided only that the periods of the perturbed and perturbing planets are not commensurable. Even if it has not been expressly stated by anyone before now, this elegant theorem can in any event be easily proven from the principles of physical astronomy. Hence the following problem arises, which is worthy of interest as much on its own account as on account of the various artifices which its solution requires: to determine exactly the attraction of a planetary orbit or, better said, of an elliptical ring, on a point at an arbitrarily given location, where the thickness of the ring is infinitely small and variable according to the law just laid out.

2.

Denoting the eccentricity of the orbit by e and the eccentric anomaly of an arbitrary point on the orbit by E, its [differential] element $\mathrm{d}E$ will correspond to the [differential] element $(1 - e\cos E)\mathrm{d}E$ of the mean anomaly; hence the mass element that corresponds to the orbital sections which correspond to those elements is proportional to the total mass, which we will take as unity, as $(1 -$

$e\cos E)\mathrm{d}E$ is to 2π, where π signifies the semicircumference of the circle of radius 1. Hence, setting the distance of the attracted point from the point on the orbit $= \rho$, the attraction generated by the orbital element is

$$\frac{(1 - e\cos E)\mathrm{d}E}{2\pi\rho^2}$$

We will denote the semi-axis major by a and the semi-axis minor by b, and choose the first as abscissa, and the center of the ellipse as origin. Therefore $aa - bb$ wil be $= aaee$, the abcissa of a point on the orbit will be $= a\cos E$, and its ordinate will be $= b\sin E$. Finally, we indicate the distance of the attracted point from the orbital plane by C, and the other coordinates parallel to the two axes by A and B. This done, the attraction of an orbital element will be resolved into two components parallel to to the axes and one perpendicular to the orbital plane:

$$\frac{(A - a\cos E)(1 - e\cos E)\mathrm{d}E}{2\pi\rho^3} = \mathrm{d}\xi$$
$$\frac{(B - b\cos E)(1 - e\cos E)\mathrm{d}E}{2\pi\rho^3} = \mathrm{d}\eta$$
$$\frac{C(1 - e\cos E)\mathrm{d}E}{2\pi\rho^3} = \mathrm{d}\zeta$$

where $\rho = \sqrt{(A - a\cos E)^2 + (B - b\sin E)^2 + CC}$.

If these integrals are integrated from $E = 0$ to $E = 360^o$, the attraction components ξ, η, ζ, which comprise the total attraction, thus arise in the direction of the respective coordinate axes, and can be resolved into every other direction by known methods.

3.

The core of the matter thus lies in introducing another variable in place of E, by which the root obtains a simpler form. To this end, we set:

$$\cos E = \frac{\alpha + \alpha'\cos T + \alpha''\sin T}{\gamma + \gamma'\cos T + \gamma''\sin T}, \quad \cos E = \frac{\beta + \beta'\cos T + \beta''\sin T}{\gamma + \gamma'\cos T + \gamma''\sin T}$$

where, however, the nine coefficients α, α', α'', etc. are obviously not completely arbitrary, but must rather satisfy specific conditions which are to be investigated next. To begin with [Zuvrdest] we observe that the substitution remains the same if all coefficients are multiplied with the same factor, so that without lessening the generality, one of them can be given a definite value, e.g. γ, can be set $= 1$; nevertheless all nine magnitudes may remain undetermined for the time being on account of the better form. We observe further that such values of α, α', α'', or β, β', β'', are excluded which are proportional to γ, γ', γ'', respectively, since otherwise E no longer remains variable. Hence, the magnitudes $\gamma'\alpha'' - \alpha'\gamma''$, $\gamma''\alpha - \gamma\alpha''$ and $\gamma\alpha' - \gamma'\alpha$ may not vanish simultaneously.

Clearly, the coefficients α, α', α'', etc. must be so constituted that in general

$$\left.\begin{array}{r}(\alpha+\alpha'\cos T+\alpha''\sin T)^2\\+(\beta+\beta'\cos T+\beta''\sin T)^2\\-(\gamma+\gamma'\cos T+\gamma''\sin T)^2\end{array}\right\}=0$$

on account of which this function must necessarily have the form

$$k(\cos T^2+\sin T^2-1)$$

From this we obtain six equations of condition

$$\left.\begin{array}{rrrcr}-\alpha\alpha & -\beta\beta & +\gamma\gamma & = & k\\-\alpha'\alpha' & -\beta'\beta' & +\gamma'\gamma' & = & -k\\-\alpha''\alpha'' & -\beta''\beta'' & +\gamma''\gamma'' & = & -k\\-\alpha'\alpha'' & -\beta'\beta'' & +\gamma'\gamma'' & = & 0\\-\alpha''\alpha & -\beta''\beta & +\gamma''\gamma & = & 0\\-\alpha\alpha' & -\beta\beta' & +\gamma\gamma' & = & 0\end{array}\right\}\text{(I)}$$

From these equations follow several others, whose development is well worth the effort. If we set for brevity

$$\alpha\beta\gamma+\alpha\beta\gamma+\alpha\beta\gamma-\alpha\beta\gamma-\alpha\beta\gamma=\epsilon\,.........\,\text{(II)}$$

the following nine equations are easily derived from the combination of equations (I):

$$\left.\begin{array}{rl}\epsilon\alpha= & -k(\beta'\gamma''-\gamma'\beta'')\\\epsilon\beta= & -k(\gamma'\alpha''-\alpha'\gamma'')\\\epsilon\gamma= & +k(\alpha'\beta''-\beta'\alpha'')\\\epsilon\alpha'= & +k(\beta''\gamma-\gamma''\beta)\\\epsilon\beta'= & +k(\gamma''\alpha-\alpha''\gamma)\\\epsilon\gamma'= & -k(\alpha''\beta-\beta''\alpha)\\\epsilon\alpha''= & +k(\beta\gamma'-\gamma\beta')\\\epsilon\beta''= & +k(\gamma\alpha'-\alpha\gamma')\\\epsilon\gamma''= & -k(\alpha\beta'-\beta\alpha')\end{array}\right\}\text{(III)}$$

Again, from the first three of these equations we obtain the following

$$\begin{array}{r}\epsilon\alpha(\beta'\gamma''-\gamma'\beta'')+\epsilon\beta(\gamma'\alpha''-\alpha'\gamma'')+\epsilon\gamma(\alpha'\beta''-\beta'\alpha'')\\=-k(\beta'\gamma''-\gamma'\beta'')^2-k(\gamma'\alpha''-\alpha'\gamma'')^2+k(\alpha'\beta''-\beta'\alpha'')^2\end{array}$$

which is equivalent to

$$\epsilon\epsilon=k(-\alpha'\alpha'-\beta'\beta'+\gamma'\gamma')(-\alpha''\alpha''-\beta''\beta''+\gamma''\gamma'')-k(-\alpha'\alpha''-\beta'\beta''+\gamma'\gamma'')^2$$

which is transformed by means of equations 2, 3, and 4 from (I) into the following:

$$\epsilon\epsilon=k^3\,.........\,\text{(IV)}$$

The following equations can easily be derived from equations (I) just as easily:

$$\left.\begin{array}{l}(\beta'\gamma'' - \gamma'\beta'')^2 = -k(k - \alpha'\alpha' - \alpha''\alpha'')\\(\gamma'\alpha'' - \alpha'\gamma'')^2 = -k(k - \beta'\beta' - \beta''\beta'')\\(\alpha'\beta'' - \beta'\alpha'')^2 = +k(k + \gamma'\gamma' + \gamma''\gamma'')\\(\beta''\gamma - \gamma''\beta)^2 = +k(k + \alpha\alpha - \alpha''\alpha'')\\(\gamma''\alpha - \alpha''\gamma)^2 = +k(k + \beta\beta - \beta''\beta'')\\(\alpha''\beta - \beta''\alpha)^2 = -k(k - \gamma\gamma + \gamma''\gamma'')\\(\beta\gamma' - \gamma\beta')^2 = +k(k + \alpha\alpha - \alpha'\alpha')\\(\gamma\alpha' - \alpha\gamma')^2 = +k(k + \beta\beta - \beta'\beta')\\(\alpha\beta' - \beta\alpha')^2 = -k(k - \gamma\gamma + \gamma'\gamma')\end{array}\right\}\text{(V)}$$

As an example we give an account of the derivation of the first formula, according to whose pattern the remainder can easily be formed. Equations 4, 2, and 3, from (I) immediately yield:

$$(\gamma'\gamma'' - \beta'\beta'')^2 - (\gamma'\gamma' - \beta'\beta')(\gamma''\gamma'' - \beta''\beta'') = \alpha'\alpha'\alpha''\alpha'' - (\alpha'\alpha' - k)(\alpha''\alpha'' - k)$$

and this relation gives the first equation from (V), developed [entwickelt].

From these equations (V) we conclude that the value $k = 0$ is inadmissible in our investigation; since otherwise all nine quantities $\beta'\gamma'' - \gamma'\beta''$, etc. would necessarily vanish, i.e. the coefficients $\alpha,\ \alpha',\ \alpha''$, would turn out to be proportional to $\beta,\ \beta',\ \beta''$ and $\gamma,\ \gamma',\ \gamma''$. Therefore the value ϵ also cannot vanish on account of equation (IV), and therefore k must necessarily be a positive magnitude, if otherwise all coefficients $\alpha,\ \alpha',\ \alpha''$, etc. are to be real. If the three first equations from (III) are combined with the three first equations from (V), the following relations arise, which are obviously dependent on the non-vanishing value of k:

$$\left.\begin{array}{l}\alpha\alpha - \alpha'\alpha' - \alpha''\alpha'' = -k\\\beta\beta - \beta'\beta' - \beta''\beta'' = -k\\\gamma\gamma - \gamma'\gamma' - \gamma''\gamma'' = +k\end{array}\right\}\text{(VI)}$$

The combination of the rest would yield the same thing. To these we add the three following:

$$\left.\begin{array}{l}\beta\gamma - \beta'\gamma' - \beta''\gamma'' = 0\\\gamma\alpha - \gamma'\alpha' - \gamma''\alpha'' = 0\\\alpha\beta - \alpha'\beta' - \alpha''\beta'' = 0\end{array}\right\}\text{(VII)}$$

which are easily obtained from equations (III); e.g., the second, fifth, and eighth equations give:

$$\epsilon\beta\gamma - \epsilon\beta'\gamma' - \epsilon\beta''\gamma'' = -k\gamma(\gamma'\alpha'' - \alpha'\gamma'') - k\gamma'(\gamma''\alpha - \alpha''\gamma) - k\gamma''(\gamma\alpha' - \alpha\gamma') = 0$$

Obviously these relations are also dependent on the exclusion of the value $k = 0$. [1]

Since, as was already remarked above, all coefficients α, α', α'', etc. can be multiplied with the same factor, and in so doing multiplying the value of k by the square of this factor, we set from now on:

$$k = 1$$

from which it follows that necessarily either $\epsilon = +1$ or $\epsilon = -1$. Thus it becomes clear that the nine coefficients α, α', α'', etc., between which the six equations of condition hold, must be reducible to three mutually independent magnitudes, which will be achieved most simply by means of three angles using the following statements:

$$\begin{aligned}
\alpha &= \cos L \tan N \\
\beta &= \sin L \tan N \\
\gamma &= \sec N \\
\alpha' &= \cos L \cos M \sec N \pm \sin L \sin M \\
\beta' &= \sin L \cos M \sec N \mp \cos L \sin M \\
\gamma' &= \cos M \tan N \\
\alpha'' &= \cos L \sin M \sec N \mp \sin L \cos M \\
\beta'' &= \sin L \sin M \sec N \pm \cos L \cos M \\
\gamma'' &= \sin M \tan N
\end{aligned}$$

where the upper of the two signs is taken in the case $\epsilon = +1$, and the lower in the case $\epsilon = -1$. However, the analytic treatment in large part will be more elegantly carried out without the use of these angles. Moreover it would not be difficult to give the geometric meaning of these angles as well as the remaining auxiliary magnitudes entering into this investigation, but we forego this explanation, since. for our subject, it is not necessary to the seasoned reader.

4.

If now the values given above are substituted for $\cos E$ and $\sin E$ in the expression for the distance, ρ, it is transformed into the following form:

$$\rho = \frac{\sqrt{G + G' \cos T^2 + G'' \sin T^2 + 2H \cos T \sin T + 2H' \sin T + 2H'' \cos T}}{\gamma + \gamma' \cos T + \gamma'' \sin T}$$

where we desire to determine the the coefficients α, α', α'', etc. such that from the six equations of condition

[1] It is perhaps not superfluous to remark that we have intentionally chosen the preceding representation and preferred it to another derivation of the relations III to VII, which indeed appears to be somewhat more elegant, despite the fact that, taken rigorously, some considerations underlie which could not be accounted for without being too lengthy.

$$\left.\begin{array}{rrrcr} -\alpha\alpha & -\beta\beta & +\gamma\gamma & = & 1 \\ -\alpha'\alpha' & -\beta'\beta' & +\gamma'\gamma' & = & -1 \\ -\alpha''\alpha'' & -\beta''\beta'' & +\gamma''\gamma'' & = & -1 \\ -\alpha'\alpha'' & -\beta'\beta'' & +\gamma'\gamma'' & = & 0 \\ -\alpha''\alpha & -\beta''\beta & +\gamma''\gamma & = & 0 \\ -\alpha\alpha' & -\beta\beta' & +\gamma\gamma' & = & 0 \end{array}\right\} [1]$$

and moreover, such that

$$H = 0, \qquad H' =, \qquad H'' = 0$$

by means of which, generally speaking, the problem will be determined. Hence, if we denote the denominator of ρ by t, the function

$$(AA + BB + CC)tt + aa(t\cos E)^2 + bb(t\sin E)^2 - 2aAt \cdot t\cos E - 2bBt \cdot t\sin E$$

of the three magnitudes t, $t\cos E$, $t\sin E$ must become

$$G + G'\cos T^2 + G''\sin T^2$$

through the substitution

$$\begin{array}{ll} t\cos E & = \alpha + \alpha'\cos T + \alpha''\sin T \\ t\sin E & = \beta + \beta'\cos T + \beta''\sin T \\ t & = \gamma + \gamma'\cos T + \gamma''\sin T \end{array}$$

It is clear that this is the same as if it were required that the function (W)

$$aaxx + bbyy + (AA + BB + CC)zz - 2aAxz - 2bByz$$

of the three indeterminates x, y, z, should become the function

$$Guu + G'u'u' + G''u''u''$$

of the indeterminates $u,\ u',\ u''$, by means of the substitution

$$\begin{array}{l} x = \alpha u + \alpha' u' + \alpha'' u'' \\ y = \beta u + \beta' u' + \beta'' u'' \\ z = \gamma u + \gamma' u' + \gamma'' u'' \end{array}$$

However, since by means of [1] it easily follows from these formulas that

$$\begin{array}{ll} u & = -\alpha x - \beta y + \gamma z \\ u' & = \alpha' x + \beta' y - \gamma' z \\ u'' & = \alpha'' x + \beta'' y - \gamma'' z \end{array}$$

function W must obviously be identical with the following:

$$G(-\alpha x - \beta y + \gamma z)^2 + G'(\alpha' x + \beta' y - \gamma' z)^2 + G''(\alpha'' x + \beta'' y - \gamma'' z)^2$$

from which we obtain the six equations:

$$\left.\begin{aligned} aa &= G\alpha\alpha + G'\alpha'\alpha' + G''\alpha''\alpha'' \\ bb &= G\beta\beta + G'\beta'\beta' + G''\beta''\beta'' \\ AA + BB + CC &= G\gamma\gamma + G'\gamma'\gamma' + G''\gamma''\gamma'' \\ bB &= G\beta\gamma + G'\beta'\gamma' + G''\beta''\gamma'' \\ aA &= G\gamma\alpha + G'\gamma'\alpha' + G''\gamma''\alpha'' \\ 0 &= G\alpha\beta + G'\alpha'\beta' + G''\alpha''\beta'' \end{aligned}\right\}\ ([1])$$

From these twelve equations, [1] and [2], we will have to determine our unknowns $G,\ G',\ G'',\ \alpha,\ \alpha',\ \alpha''$, etc.

5.

By combination of the equations [1] and [2], the following are easily derived:

$$\begin{aligned} -\alpha aa + \gamma aA &= \alpha G \\ -\beta bb + \gamma bB &= \beta G \\ \gamma(AA + BB + CC) - \alpha aA - \beta bB &= \gamma G \end{aligned}$$

wherefrom it follows further that

$$\alpha = \frac{\gamma aA}{aa + G} \ldots\ldots\ldots\ldots [3]$$

$$\beta = \frac{\gamma bB}{bb + G} \ldots\ldots\ldots\ldots [4]$$

$$AA + BB + CC - \frac{aaAA}{aa + G} - \frac{bbBB}{bb + G} = G$$

We can also write the latter relation thus:

$$\frac{AA}{aa + G} + \frac{BB}{bb + G} + \frac{CC}{G} = 1 \ldots\ldots\ldots [5]$$

Further, from the combinations of equations [1] and [2], we deduce:

$$\begin{aligned} \alpha' aa - \gamma' aA &= \alpha' G' \\ \beta' bb - \gamma' bB &= \beta' G' \\ -\gamma(AA + BB + CC) - \alpha' aA - \beta' bB &= \gamma' G' \end{aligned}$$

and from that

$$\alpha' = \frac{\gamma' aA}{aa - G'} \ldots\ldots\ldots\ldots [6]$$

$$\beta' = \frac{\gamma' bB}{bb - G'} \ldots\ldots\ldots\ldots [7]$$

$$\frac{AA}{aa - G'} + \frac{BB}{bb - G'} - \frac{CC}{G'} = 1 \ldots\ldots\ldots [8]$$

and finally, in the same way:

$$\alpha'' = \frac{\gamma'' aA}{aa - G''} \ldots\ldots\ldots\ldots [9]$$

$$\beta'' = \frac{\gamma'' bB}{bb - G''} \ldots\ldots\ldots\ldots [10]$$

$$\frac{AA}{aa - G''} + \frac{BB}{bb - G''} - \frac{CC}{G''} = 1 \ldots\ldots\ldots [11]$$

Hence, it can be seen that G, $-G'$, $-G''$ are the roots of the equation

$$\frac{AA}{aa + x} + \frac{BB}{bb + x} - \frac{CC}{x} = 1 \ldots\ldots\ldots [12]$$

which, developed in terms of powers of x, has the form [13]:

$$x^3 - (AA + BB + CC - aa - bb)xx + (aabb - aaBB - aaCC - bbAA - bbCC)x - aabbCC$$

$$= 0$$

6.

Now, from the nature of this cubic equation, the following can be immediately inferred:

I. From the last term of the equation $-aabbCC$ it is concluded that this surely has a real root which is either positive or, in the case $C = 0$, equal to zero. We will denote this real, non-negative root by g.

II. Putting equation [12] in the following form:

$$x = \frac{AAx}{aa + x} + \frac{BBx}{bb + x} + CC$$

and subtracting

$$g = \frac{AAg}{aa + g} + \frac{BBg}{bb + g} + CC$$

and dividing by $x - g$, a new equation thus arises,

$$1 = \frac{aaAA}{(aa + x)(aa + g)} + \frac{bbBB}{bb + x} bb + g$$

which yields the two remaining roots, and which, properly organized and solved, gives [14]:

$$2x =$$

$$\frac{aaAA}{aa + g} + bbBBbb + g - aa - bb \pm \sqrt{(aa - bb - \frac{aaAA}{aa + g} + \frac{bbBB}{bb + g})^2 + \frac{4aabbAABB}{(aa + g)(bb + g)}}$$

Since the magnitude under the root is by its nature positive, or at least nonnegative, this expression shows that the remaining two roots also always turn out real.

III. However, if the same equations are subtracted from one another in the following form:

$$\begin{aligned} gx &= \tfrac{AAgx}{aa+x} + \tfrac{BBgx}{bb+x} + gCC \\ gx &= \tfrac{AAgx}{aa+g} + \tfrac{BBgx}{bb+g} + xCC \end{aligned}$$

and divided by $g - x$, produces an equation for the two remaining roots, contained in the form:

$$0 = \frac{AAgx}{(aa+g)(aa+x)} + \frac{BBgx}{(bb+g)(bb+x)} + CC$$

which clearly, if g is a positive magnitude, cannot be satisfied by a positive value of x, wherefrom we conclude that our cubic equation cannot have more than one positive root.

IV. Hence, so long as 0 is not among the roots of our equation, there will necessarily be one positive and two negative roots. However, if $C = 0$, and thus 0 is one of the roots, the remaining roots will be yielded by the equation

$$xx - (AA + BB - aa - bb)x + aabb - aaBB - bbAA = 0$$

whence these roots are expressed by

$$\frac{1}{2}(AA + BB - aa - bb) \pm \frac{1}{2}\sqrt{(AA - BB - aa + bb)^2 + 4AABB}$$

Here, we will again have to distinguish three cases:

First, if the last term $aabb - aaBB - bbAA$ is positive (i.e., if the attracted point falls *within* the curve of the attracting ellipse, in the same plane), both roots, since they must be real, will have the same sign, and since they cannot both be positive simultaneously, they will necessarily be negative. Moreover, it can be concluded from the data, independently of what was already proven, that the middle coefficient, which can be put in the form

$$(aabb - aaBB - bbAA)(\frac{1}{aa} + \frac{1}{bb}) + \frac{bbAA}{aa} + \frac{aaBB}{bb}$$

is clearly positive in this case.

Second, if the last term is negative, the attracted point therefore lies *outside* the ellipse, in the plane of the ellipse, and thus one root is necessarily positive, and the other negative.

Third, however, if the last term vanishes, the attracted point then lies in the periphery of the ellipse itself, and the second root will also be $= 0$, while the third

$$= -\frac{bbAA}{aa} - \frac{aaBB}{bb}$$

and is thus negative. In what remains, this physically impossible case, in which the attraction itself turns out to be infinitely large, will be excluded from our investigation, at least in this location.

7.

For the determination of the coefficients $\gamma,\ \gamma',\ \gamma''$, we find from the equations [1], [3], [4], [6], [7], [9], [10]:

$$\left.\begin{aligned}\gamma &= \frac{1}{\sqrt{1-(\frac{aA}{aa+G})^2-(\frac{bB}{bb+G})^2}}\\ \gamma' &= \frac{1}{\sqrt{(\frac{aA}{aa-G'})^2+(\frac{bB}{bb-G'})^2-1}}\\ \gamma'' &= \frac{1}{\sqrt{(\frac{aA}{aa-G''})^2+(\frac{bB}{bb-G''})^2-1}}\end{aligned}\right\}[15]$$

If these equations are correctly combined with (5), (8), and (11), it follows that:

$$\left.\begin{aligned}\gamma &= \frac{G}{\sqrt{(\frac{AG}{aa+G})^2+(\frac{BG}{bb+G})^2+CC}}\\ \gamma' &= \frac{G'}{\sqrt{(\frac{AG'}{aa-G'})^2+(\frac{BG'}{bb-G'})^2+CC}}\\ \gamma'' &= \frac{G''}{\sqrt{(\frac{AG''}{aa-G''})^2+(\frac{BG''}{bb-G''})^2+CC}}\end{aligned}\right\}[16]$$

These last expressions show that none of the quantities $G,\ G',\ G''$, can be negative if $\gamma,\ \gamma',\ \gamma''$ are to be real.

In the case that C is not equal to zero, G must necessarily be set equal to the positive root of equation [13], and hence it becomes clear that $-G'$ must be equal to one negative root, and $-G''$ equal to the other; [2] which root we take for $-G'$ and which for $-G''$ will be wholly arbitrary.

In the case where $C = 0$ and the attracted point lies inside the curve, the two negative roots of equation 13 must necessarily be taken for $-G'$ and $-G''$, and therefore G must be set $= 0$. However, since in this case the first formula in 16 becomes indeterminate, we utilize in the first formula from [15] instead, which yields:

$$\gamma = \frac{1}{\sqrt{1 - \frac{AA}{aa} - \frac{BB}{bb}}}$$

However, in case $C = 0$, where the attracted point lies outside of the ellipse, the positive root of the equation is set equal to G, and either the negative $= -G'$ and $G'' = 0$, or the negative root $= -G''$ and $G' = 0$; we then find the coefficients γ'' or γ' from the formula

$$\frac{1}{\sqrt{\frac{AA}{aa} + \frac{BB}{bb}}}$$

Finally, in the cases we just excluded, where the attracted point is taken as on the peripherie of the ellipse itself, the coefficients γ and γ' or γ and γ'' turn

[2]Actually, it follows from the preceding analysis only that $-G'$ and $-G''$ must satisfy equation 13, whence it still seems doubtful whether it is not to be allowed to set $-G'$ as well as $-G''$ equal to the *same* negative root, while the third is wholly neglected. But it is easily recognized that if the second and third roots of the equation are different, it would follow from $-G' = -G''$ that $\gamma' = \gamma''$, $\alpha' = \alpha''$, $\beta' = \beta''$, and thus $-\alpha'\alpha'' - \beta'\beta'' + \gamma'\gamma'' = -\alpha'\alpha' - \beta'\beta' + \gamma'\gamma' = 1$, which contradicts the fourth equation in [I]. cf. what will be said in what follows about the case of two equal roots of equation 13.

out to be infinitely large, which means that our transformation is altogether inapplicable in this case.

8.

Although the formulas 15 and 16 are sufficient for the determination of the coefficients $\gamma,\ \gamma',\ \gamma''$, we can in fact replace them with more elegant ones. To this end we multiply equation 5 by $aabb-GG$, from which follows after a simple reduction:

$$\frac{aaAA(bb+G)}{aa+G}-AAG+\frac{bbBB(aa+G)}{bb+G}-BBG+\frac{aabbCC}{G}-CCG=aabb-GG$$

It follows, however, from the nature of the cubic equation that
the sum of the roots $G-G'-G''=AA+BB+CC-aa-bb$, and
the product of the roots $GG'G''=aabbCC$. Hence, the foregoing equation is transformed into the following form:

$$\frac{aaAA(bb+G)}{aa+G}+\frac{bbBB(aa+G)}{bb+G}+G'G''-G(G-G'-G''+aa+bb)=aabb-GG$$

which can also be represented thus:

$$\frac{aaAA(bb+G)}{aa+G}+\frac{bbBB(aa+G)}{bb+G}-(aa+G)(bb+G)+(G+G')(G+G'')=0$$

The value of the coefficients γ is transformed by the first formula in [15] into the following:

$$\gamma=\sqrt{\frac{(aa+G)(bb+G)}{(G+G')(G+G'')}}\ \ldots\ldots\ldots[17]$$

By a completely similar analysis we obtain:

$$\gamma'=\sqrt{\frac{(aa-G')(bb-G')}{(G+G')(G''-G')}}\ \ldots\ldots\ldots[18]$$

$$\gamma''=\sqrt{\frac{(aa-G'')(bb-G'')}{(G+G'')(G'-G'')}}\ \ldots\ldots\ldots[19]$$

After the determination of the coefficients $\gamma,\ \gamma',\ \gamma''$, the remaining, $\alpha,\ \beta,\ \alpha',\ \beta',\ \alpha'',\ \beta''$ can be derived from them by means of the formulas 3, 4, 6, 7, 9, 10.

9.

It is easily recognized that the signs of the roots through which we have determined $\gamma,\ \gamma',\ \gamma''$, can be taken arbitrarily. However it is worth the effort to investigate in what way the sign of the value ϵ is related to those signs. To this end we consider the third equation in III, Art. 3.

$$\epsilon\gamma=\alpha'\beta''-\beta'\alpha''$$

which is transformed by means of the formulas 6, 7, 9, 10 into:

$$\begin{aligned}\epsilon\gamma &= \frac{abAB\gamma'\gamma''}{(a^2-G')(b^2-G'')} - \frac{abAB\gamma'\gamma''}{(a^2-G'')(b^2-G')} \\ &= \frac{ab(a^2-b^2)AB(G''-G')\gamma'\gamma''}{(a^2-G')(a^2-G'')(b^2-G')(b^2-G'')}\end{aligned}$$

But from consideration of equation 13 we easily deduce:

$$\begin{aligned}(aa+G)(aa-G')(aa-G'') &= aa(aa-bb)AA \\ (bb+G)(bb-G')(bb-G'') &= -bb(aa-bb)BB\end{aligned}$$

And so the preceding equation becomes:

$$\epsilon\gamma = \frac{(aa+G)(bb+G)(G'-G'')\gamma'\gamma''}{ab(aa-bb)AB}$$

which, combined with 17, yields:

$$\gamma\gamma'\gamma'' = \frac{\epsilon ab(aa-bb)AB}{(G+G')(G+G'')(G'-G'')}$$

From that it becomes clear that if the negative root of the cubic equation with the largest absolute value is taken, and at the same time the coefficients γ, γ', γ'', are all taken as positive, ϵ receives the same sign as AB, and that the same thing happens if either all four or only two of these four conditions are not fulfilled, and that the opposite happens, however, if one or three of the conditions are violated. It is convenient, moreover, to take down here the following relations, which can be easily derived from the preceding:

$$\begin{aligned}\alpha\alpha'\alpha'' &= \frac{\epsilon aabAAB}{(G+G')(G+G'')(G'-G'')} \\ \beta\beta'\beta'' &= -\frac{\epsilon abbABB}{(G+G')(G+G'')(G'-G'')} \\ \alpha\beta &= \frac{abAB}{(G+G')(G+G'')} \\ \alpha'\beta' &= -\frac{abAB}{(G+G')(G-G'')} \\ \alpha''\beta'' &= \frac{abAB}{(G'+G'')(G'-G'')}\end{aligned}$$

10.

Our formulas can become indeterminate in certain cases, which must be considered separately. First, we want to deal with the case that the negative roots $-G'$, G'', of the cubic equation become equal, whereby the coefficients γ, γ'', appear to assume infinite values according to the formulas 18 and 19, but are actually indeterminate.

If g is set $= G$ in formula 14, so that two values of x, namely $-G'$ and $-G''$, coincide, then we must have

$$AB = 0, \quad aa - bb - \frac{aaAA}{aa+G} + \frac{bbBB}{bb+G} = 0$$

It is easily recognized from that, that since $aa - bb$ by its nature is either positive or equal to zero, it must be the case that

$$B = 0, \text{ and}$$

$$\text{either } aa - bb = \frac{aaAA}{aa+G}, \quad \text{or} \quad aa + G = \frac{aaAA}{aa-bb}$$

Substituting these values into 14, we obtain:

$$G' = G'' = bb.$$

If, further, the value of $x = -bb$ is entered in the cubic equation 13, it thus arises that:

$$(aa - bb)(CC + bb) = bbAA$$

When this equation of condition occurs at the same time as $B = 0$, we are presented with the case under consideration. And since

$$G = \frac{aaAA}{aa-bb} - aa = \frac{aaCC}{bb}$$

formula 17 yields

$$\gamma = \sqrt{\frac{aabbAA}{(aa-bb)(aaCC+b^4)}} = \sqrt{\frac{aaCC+aabb}{aaCC+b^4}}$$

and then formulas 3 and 4:

$$\begin{aligned}\alpha &= \tfrac{\gamma(aa-bb)}{aA} = \tfrac{\gamma bbA}{a(CC+bb)} = \sqrt{\tfrac{bb(aa-bb)}{aaCC+b^4}} = \sqrt{\tfrac{b^4AA}{(CC+bb)(aaCC+b^4)}} \\ \beta &= 0\end{aligned}$$

The values of coefficients γ', γ'', obtained by means of 18 and 19, remain indeterminate in this case, as do the values of the remaining coefficients α', β', α'', β''. Nevertheless, any one of the coefficients can express the remaining five. For example, as per formula 6, let

$$\alpha' = \frac{\gamma' aA}{aa-bb}$$

and further:

$$\beta' = \sqrt{1 - \alpha'\alpha' + \gamma'\gamma'}, \quad \gamma'' = \sqrt{\gamma\gamma - 1 - \gamma'\gamma'}, \quad \beta'' = \sqrt{a\alpha''\alpha'' + \gamma''\gamma''}$$

However, it is more elegantly accomplished in the following manner. From

$$\gamma\gamma = 1 + \alpha\alpha, \quad \alpha\alpha' = \gamma\gamma', \quad 1 = \alpha'\alpha' + \beta'\beta' - \gamma'\gamma'$$

follows

$$\beta'\beta' + \frac{\gamma'\gamma'}{\alpha\alpha} = 1 - \alpha'\alpha' + \frac{\gamma\gamma\gamma'\gamma'}{\alpha\alpha} = 1.$$

Hence, we can set:

$$\beta' = \cos f, \quad \gamma' = \alpha \sin f, \quad \alpha' = \gamma \sin f$$

Then, however, from the formula

$$\epsilon\alpha'' = \beta\gamma' - \gamma\beta', \quad \epsilon\beta'' = \gamma\alpha' - \alpha\gamma', \quad \epsilon\gamma'' = \beta\alpha' - \alpha\beta', \quad \epsilon\epsilon = 1$$

we find

$$\alpha'' = -\epsilon\gamma\cos f, \quad \beta'' = \epsilon\sin f, \quad \gamma'' = -\epsilon\alpha\cos f$$

The value of the angle f is arbitrary here, and ϵ can also be chosen arbitrarily as $= +1$, or $= -1$.

11.

If G', G'', are unequal, the values for the coefficients γ, γ', γ'', by formulas 17, 18, 19, cannot be indeterminate; on the other hand, as soon as one of the magnitudes $aa - G'$, $bb - G'$, $aa - G''$, $bb - G''$ vanishes, the values of the coefficients α', β', α'', γ'' derived from formulas 6, 7, 9, and 10 respectively, at first glance appear to become indeterminate; a little attentiveness, however, shows that the case is otherwise. For example, if we assume that $aa - G' = 0$, then $\gamma' = 0$ by equation 18 and $\beta' = 0$ by equation 7 (provided it were not simultaneously the case that $aa = bb$), according to which it must necessarily be the case that $\alpha' = \pm 1$. If, however, $aa = bb$, the formula which precedes equation (6) of article 5 yields $\alpha' A + \beta' B = 0$, which, combined with $\alpha'\alpha' + \beta'\beta' = 1$ produces

$$\alpha' = \frac{B}{\sqrt{AA + BB}}, \quad \beta' = \frac{-A}{\sqrt{AA + BB}}$$

These expressions can obviously not be indeterminate, unless A and B were simultaneously $= 0$; but then we would have returned to the case already treated in the preceding article.

12.

Now that we have shown the way to completely determine the twelve magnitudes G, G', G'', α, α', α'', β, β', β'', γ, γ', γ'', we move on to the development of the differential $\mathrm{d}E$. We set

$$t = \gamma + \gamma'\cos T + \gamma''\sin T \ldots\ldots\ldots\ldots [20]$$

so that

$$t\cos E = \alpha + \alpha'\cos T + \alpha''\sin T \ldots\ldots\ldots\ldots [21]$$
$$t\sin E = \beta + \beta'\cos T + \beta''\sin T \ldots\ldots\ldots\ldots [22]$$

From which we conclude:

$$\begin{aligned} t\mathrm{d}E &= \cos E\cdot \mathrm{d}(\sin E) - \sin E\cdot \mathrm{d}(t\cos E) \\ &= \cos E(\beta''\cos T - \beta'\sin T)\mathrm{d}T - \sin E(\alpha''\cos T - \alpha'\sin T)\mathrm{d}T \end{aligned}$$

and hence:

$$\begin{aligned} tt\mathrm{d}E &= (\alpha\beta'' - \alpha''\beta)\cos T\cdot \mathrm{d}T + (\alpha'\beta - \beta'\alpha)\sin T\cdot \mathrm{d}T + (\alpha'\beta'' - \beta'\alpha'')\mathrm{d}T \\ &= \epsilon\gamma'\cos T\mathrm{d}T + \epsilon\gamma''\sin T\mathrm{d}T + \epsilon\gamma\mathrm{d}T = \epsilon t\mathrm{d}T \end{aligned}$$

or

$$t\mathrm{d}E = \epsilon \mathrm{d}T \ldots\ldots\ldots\ldots [23]$$

It is worth noting that the quantity t, by its nature, is always positive if the coefficient γ is positive, and negative if γ is negative. Namely, since

$$(\gamma' \cos T + \gamma'' \sin T)^2 + (\gamma'' \cos T - \gamma' \sin T)^2 = \gamma'\gamma' + \gamma''\gamma'' = \gamma\gamma - 1,$$

it is perceived that, irrespective of sign, $\gamma' \cos T + \gamma'' \sin T$ will always be less than γ. From that we conclude that if $\epsilon\gamma$ is a positive quantity, the variables E and T constantly grow together; however, if $\epsilon\gamma$ is negative, the one variable continuously decreases while the other grows.

13.

The connection between the variables E and T will be better explained through the following consideration. If we set $\sqrt{\gamma\gamma - 1} = \delta$ so that $\delta\delta = \alpha\alpha + \beta\beta = \gamma'\gamma' + \gamma''\gamma''$, we conclude from equations 20, 21, 22, that:

$$\begin{aligned} t(\delta + \alpha \cos E + \beta \sin E) &= \gamma\delta + \alpha\alpha + \beta\beta + (\gamma'\delta + \alpha\alpha' + \beta\beta') \cos T \\ &\quad + (\gamma''\delta + \alpha\alpha'' + \beta\beta'') \sin T \\ &= (\gamma + \delta)(\delta + \gamma' \cos T + \gamma'' \sin T) \end{aligned}$$

It further follows from equations 21, 22, that

$$t(\alpha \sin E - \beta \cos E) = \epsilon(\gamma' \sin T - \gamma'' \cos T)$$

If we set

$$\frac{\alpha}{\delta} = \cos L, \quad \frac{\beta}{\delta} = \sin L, \quad \frac{\gamma'}{\delta} = \cos M, \quad \frac{\gamma''}{\delta} = \sin M$$

these equations assume the following form:

$$\begin{aligned} t(1 + \cos(E - L)) &= (\gamma + \delta)(1 + \cos(T - M)) \\ t \sin(E - L) &= \epsilon \sin(T - M) \end{aligned}$$

from which follows, through division and on account of the fact that $(\gamma + \delta)(\gamma - \delta) = 1$:

$$\begin{aligned} \tan \tfrac{1}{2}(E - L) &= \epsilon(\gamma - \delta) \tan \tfrac{1}{2}(T - M) \\ \tan \tfrac{1}{2}(T - M) &= \epsilon(\gamma + \delta) \tan \tfrac{1}{2}(E - L) \end{aligned}$$

From this is yielded not only the conclusion to which we were led at the end of the foregoing article, but rather it also becomes clear besides that, if the value of E is increased by 360 degrees, the value of T grows or decreases by the same amount, depending on whether $\epsilon\gamma$ is positive or negative. If we further set $\delta = \tan N$ and $\gamma = \sec N$, then clearly

$$\gamma - \delta = \tan(45^o - \frac{1}{2}N), \quad \gamma + \delta = \tan(45^o + \frac{1}{2}N)$$

14.

From the combination of the equations 20, 21, 22 with those of article 5, we obtain:

$$\begin{aligned} at(A - a\cos E) &= \alpha G - \alpha' G' \cos T - \alpha'' G'' \sin T \\ bt(B - b\sin E) &= \beta G - \beta' G' \cos T \beta'' G'' \sin T \end{aligned}$$

Hence, setting, for brevity:

$$\begin{aligned} (\alpha G - \alpha' G' \cos T - \alpha'' G'' \sin T)(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T) = aX \\ (\beta G - \beta' G' \cos T - \beta'' G'' \sin T)(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T) = bY \\ C(\gamma + \gamma' \cos T + \gamma'' \sin T)(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T) = Z \end{aligned}$$

we have

$$\mathrm{d}\xi = \frac{\epsilon X \mathrm{d}T}{2\pi t^3 \rho^3}, \quad \mathrm{d}\eta = \frac{\epsilon Y \mathrm{d}T}{2\pi t^3 \rho^3}, \quad \mathrm{d}\zeta = \frac{\epsilon Z \mathrm{d}T}{2\pi t^3 \rho^3}$$

However, we have

$$t\rho = \pm\sqrt{G + G' \cos T^2 + G'' \sin T^2}$$

wherein the upper and lower sign apply according as t is a positive or negative magnitude, respectively, (that is to say that ρ is, by its nature, always taken as positive), that is, according as the coefficient γ is positive or negative. Hence

$$\frac{\epsilon \mathrm{d}T}{2\pi t^3 \rho^3} = \pm \frac{\mathrm{d}T}{2\pi(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$

where the undetermined sign is dependent on the sign of $\gamma\epsilon$.

In order to now find the values of ξ, η, ζ, themselves, the integration of the differential must be carried out, from the value T which corresponds to $E = 0$, to the one which corresponds to $E = 360^o$, or also (which obviously amounts to the same thing), from a value of T which corresponds to an arbitrary value of E, to the value of T which corresponds to an E which is increased by 360^o; the integration can therefore extend from $T = 0$ to $T = 360^o$, if $\epsilon\gamma$ is a positive magnitude, or from $T = 360^o$ until $T = 0$, if $\epsilon\gamma$ is negative. Therefore we will obviously have, independent of the sign of $\epsilon\gamma$:

$$\begin{aligned} \xi &= \int \frac{X \mathrm{d}T}{2\pi(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}} \\ \eta &= \int \frac{Y \mathrm{d}T}{2\pi(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}} \\ \zeta &= \int \frac{Z \mathrm{d}T}{2\pi(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}} \end{aligned}$$

where the integration extends from $T = 0$ to $T = 360^o$.

15.

It can be seen without any effort, that the integrals

$$\xi = \int \frac{\cos T \mathrm{d}T}{(G+G'\cos T^2+G''\sin T^2)^{\frac{3}{2}}}$$
$$\eta = \int \frac{\sin T \mathrm{d}T}{2\pi(G+G'\cos T^2+G''\sin T^2)^{\frac{3}{2}}}$$
$$\zeta = \int \frac{\cos T \sin T \mathrm{d}T}{2\pi(G+G'\cos T^2+G''\sin T^2)^{\frac{3}{2}}}$$

extended from $T = 180^o$ to $T = 360^o$, assume the same values as those extended from $T = 0$ to $T = 180^o$, but with the opposite sign; therefore these integrals, extended from $T = 0$ to $T = 360^o$ will obviously be equal to zero. From this we conclude, that

$$\xi = \int \frac{((\gamma-e\alpha)\alpha G-(\gamma'-e\alpha')\alpha' G'\cos T^2-(\gamma''-e\alpha'')\alpha'' G''\sin T^2)\mathrm{d}T}{2\pi a(G+G'\cos T^2+G''\sin T^2)^{\frac{3}{2}}}$$
$$\eta = \int \frac{((\gamma-e\alpha)\beta G-(\gamma'-e\alpha')\beta' G'\cos T^2-(\gamma''-e\alpha'')\beta'' G''\sin T^2)\mathrm{d}T}{2\pi b(G+G'\cos T^2+G''\sin T^2)^{\frac{3}{2}}}$$
$$\zeta = \int \frac{((\gamma-e\alpha)\gamma G-(\gamma'-e\alpha')\gamma' G'\cos T^2-(\gamma''-e\alpha'')\gamma'' G''\sin T^2)C\mathrm{d}T}{2\pi(G+G'\cos T^2+G''\sin T^2)^{\frac{3}{2}}}$$

where the integral is extended from $T = 0$ to $T = 360^o$. Therefore, if we denote the value of the integrals

$$\int \frac{\cos T^2 \mathrm{d}T}{2\pi((G+G')\cos T^2+(G+G'')\sin T^2)^{\frac{3}{2}}}$$
$$\int \frac{\sin T^2 \mathrm{d}T}{2\pi((G+G')\cos T^2+(G+G'')\sin T^2)^{\frac{3}{2}}}$$

taken over the same interval, by P, Q, we obtain:

$$\begin{aligned} a\xi &= ((\gamma-e\alpha)\alpha G-(\gamma'-e\alpha')\alpha' G')P+((\gamma-e\alpha)\alpha G-(\gamma''-e\alpha'')\alpha'' G'')Q \\ b\eta &= ((\gamma-e\alpha)\beta G-(\gamma'-e\alpha')\beta' G')P+((\gamma-e\alpha)\beta G-(\gamma''-e\alpha'')\beta'' G'')Q \\ \zeta &= ((\gamma-e\alpha)\gamma G-(\gamma'-e\alpha')\gamma' G')P+((\gamma-e\alpha)\gamma G-(\gamma''-e\alpha'')\gamma'' G'')CQ \end{aligned}$$

wherewith our problem is completely solved.

16.

Now, concerning P, Q, both will obviously become equal to

$$\frac{1}{2(G+G')^{\frac{3}{2}}}$$

when $G' = G''$, while they lead to transcendentals in every other case. The way in which these can then be expressed through series is sufficiently well known; however we hope to do the reader a kindness if we take this opportunity to develop an operation for the determination of these and other transcendentals through a particular, very handy algorithm, which we have already often been applying for many years, and of which we intend to treat more fully in other locations. Let m, n, be two positive magnitudes, and let us set

$$m' = \frac{1}{2}(m+n), \quad n' = \sqrt{mn}$$

so that m', n', are, respectively, the arithmetic and geometric means between m and n4. We assume that the geometric mean is always taken as positive. Further, let

$$m'' = \tfrac{1}{2}(m' + n'), \qquad n'' = \sqrt{m'n'}$$
$$m''' = \tfrac{1}{2}(m'' + n''), \quad n''' = \sqrt{m''n''}$$

and so forth, then the series $m,\ m',\ m'',\ m'''$, etc. and $n,\ n',\ n'',\ n'''$, etc. tend very rapidly towards a *common limit*, which we will denote by μ and which we will call, simply, the *arithmetic-geometric mean* between m and n. We will now show that $\frac{1}{\mu}$ is the value of the integral

$$\int \frac{\mathrm{d}T}{2\pi\sqrt{mm\cos T^2 + nn\sin T^2}}$$

taken from $T = 0$ to $T = 360^o$. *Proof.* We assume the variable T to be replaced by another, T', in such a way that

$$\sin T = \frac{2m\sin T'}{(m+n)\cos T'^2 + 2m\sin T'^2}$$

then it is easily recognized that, if T' grows from 0 to 90^o, 180^o, 270^o, and 360^o, T also (although in unequal intervals) grows from 0 to 90^o, 180^o, 270^o, and 360^o. However, by a proper development of the equation, it is found that

$$\frac{\mathrm{d}T}{\sqrt{mm\cos T^2 + nn\sin T^2}} = \frac{\mathrm{d}T'}{\sqrt{m'm'\cos T'^2 + n'n'\sin T'^2}}$$

and hence, the values of the integrals

$$\int \frac{\mathrm{d}T}{2\pi\sqrt{mm\cos T^2 + nn\sin T^2}}, \quad \int \frac{\mathrm{d}T'}{2\pi\sqrt{m'm'\cos T'^2 + n'n'\sin T'^2}}$$

are equal to one another, if both variables run from 0 to 360^o. And since it is clear that this can be continued in such a way that these values are also equal to the value of the integral

$$\int \frac{\mathrm{d}\theta}{2\pi\sqrt{\mu\mu\cos\theta^2 + \mu\mu\sin\theta^2}}$$

extending from $\theta = 0$ to $\theta = 360^o$, which is manifestly $= \frac{1}{\mu}$, Q.E.D.

17.

From the equation which establishes the relation between T and T',

$$(m-n)\sin T\cdot\sin T'^2 = 2m\sin T' - (m+n)\sin T$$

is easily obtained:

$$\begin{aligned} \sqrt{mm\cos T^2 + nn\sin T^2} &= m - (m-n)\sin T\cdot\sin T' \\ \sqrt{m'm'\cos T'^2 + n'n'\sin T'^2} &= m\cot T\tan T' \end{aligned}$$

and from this, by aid the same equation,

$$\begin{array}{rl} & \sin T\cdot\sin T'\cdot\sqrt{mm\cos T^2+nn\sin T^2}+m'(\cos T^2-\sin T^2)\\ = & \cos T\cdot\cos T'\cdot\sqrt{m'm'\cos T'^2+n'n'\sin T'^2}-\frac{1}{2}(m-n)\sin T'^2\end{array}$$

This equation, multiplied by

$$\frac{\mathrm{d}T}{\sqrt{mm\cos T^2+nn\sin T^2}}=\frac{\mathrm{d}T'}{\sqrt{m'm'\cos T'^2+n'n'\sin T'^2}}$$

produces

$$\frac{m'(\cos T^2-\sin T^2)\mathrm{d}T}{\sqrt{mm\cos T^2+nn\sin T^2}}=-\frac{\frac{1}{2}(m-n)\sin T'^2\mathrm{d}T'}{\sqrt{m'm'\cos T'^2+n'n'\sin T'^2}}+\mathrm{d}(\sin T'\cos T)$$

If we multiply these equations by $\frac{m-n}{\pi}$, substituting

$$\begin{array}{rcl} m'(m-n) & = & \frac{1}{2}(mm-nn)\\ (m-n)^2 & = & 4(m'm'-n'n')\\ \sin T'^2 & = & \frac{1}{2}-\frac{1}{2}(\cos T'^2-\sin T'^2)\end{array}$$

and integrate with respect to T and T' from 0 to 360^o, we thereby obtain:

$$(mm-nn)\int\frac{(\cos T^2-\sin T^2)\cdot\mathrm{d}T}{2\pi\sqrt{mm\cos T^2+nn\sin T^2}}$$

$$=-\frac{2(m'm'-n'n')}{\mu}+2(m'm'-n'n')\int\frac{(\cos T'^2-\sin T'^2)\cdot\mathrm{d}T'}{2\pi\sqrt{m'm'\cos T'^2+n'n'\sin T'^2}}$$

And since the definite integral on the right hand side can be further transformed in just the same way, the integral

$$\int\frac{(\cos T^2-\sin T^2)\cdot\mathrm{d}T}{2\pi\sqrt{mm\cos T^2+nn\sin T^2}}$$

is obviously represented by the rapidly converging infinite series

$$-\frac{2(m'm'-n'n')+4(m''m''-n''n'')+8(m'''m'''-n'''n''')+\text{etc.}}{(mm-nn)\mu}=-\frac{\nu}{\mu}$$

The numerical calculation can be carried out conveniently using logarithms, if we set

$$\frac{1}{4}\sqrt{mm-nn}=\lambda,\quad \frac{1}{4}\sqrt{m'm'-n'n'}=\lambda,\quad \frac{1}{4}\sqrt{m''m''-n''n''}=\lambda'',\quad \text{etc.}$$

whence it follows that:

$$\lambda'=\frac{\lambda\lambda}{m'},\quad \lambda''=\frac{\lambda'\lambda'}{m''},\quad \lambda'''=\frac{\lambda''\lambda''}{m'''},\quad \text{etc.}$$

and

$$\nu=\frac{2\lambda'\lambda'+4\lambda''\lambda''+8\lambda'''\lambda'''+\text{etc.}}{\lambda\lambda}$$

18.

By means of the method just explained, the *indefinite* integrals (which begin with the value of the variable $= 0$) can also be determined with the greatest elegance. Namely, T'' can be determined from m', n', T' just as T' was determined from m, n, T, and in the same way T''' from m'', n'', T'', etc. Thus, the values of the terms of the series T, T', T'', T''', etc. converge very rapidly upon a limit, θ, and we will have

$$\begin{aligned} \int \frac{\mathrm{d}T}{\sqrt{mm\cos T^2+nn\sin T^2}} &= \frac{\theta}{\mu} \\ \int \frac{(\cos T^2-\sin T^2)\mathrm{d}T}{\sqrt{mm\cos T^2+nn\sin T^2}} &= -\frac{\nu\theta}{\mu} + \frac{\lambda'\cos T\sin T'+2\lambda''\cos T'\sin T''+4\lambda'''\cos T''\sin T'''+\text{etc.}}{\lambda\lambda} \end{aligned}$$

However, it is sufficient to have indicated this here in passing, since it is not necessary for our goal.

19.

Since we now set $m = \sqrt{G+G'}$, $n = \sqrt{G+G''}$, the magnitudes P, Q, can easily be reduced to the transcendentals μ, ν. Since, namely, P, Q, are the values of the integrals

$$\int \frac{\cos T^2 \cdot \mathrm{d}}{2\pi(mm\cos T^2+nn\sin T^2)^{\frac{3}{2}}}$$

taken from $T = 0$ to $T = 360^o$, it can be seen immediately to begin with, that

$$mmP + nnQ = \frac{1}{\mu} \ldots\ldots\ldots\ldots [24]$$

Further:

$$\begin{aligned} \frac{(\cos T^2-\sin T^2)\mathrm{d}T}{2\pi\sqrt{mm\cos T^2+nn\sin T^2}} + \frac{(mm\cos T^2-nn\sin T^2)\mathrm{d}T}{2\pi\sqrt{mm\cos T^2+nn\sin T^2}^{\frac{3}{2}}} &= \frac{(mm\cos T^4-nn\sin T^4)\mathrm{d}T}{2\pi\sqrt{mm\cos T^2+nn\sin T^2}^{\frac{3}{2}}} \\ &= \mathrm{d}\cdot\frac{\cos T\sin T}{\pi\sqrt{mm\cos T^2+nn\sin T^2}} \end{aligned}$$

If these relations are integrated from $T = 0$ to $T = 360^o$, it thus follows:

$$-\frac{\nu}{\mu} + mmP - nnQ = 0 \ldots\ldots\ldots\ldots [25]$$

From the combination of equations 24, 25, we finally obtain:

$$P = \frac{1+\nu}{2mm\mu}, \quad Q = \frac{1-\nu}{2nn\mu}$$