Excerpt From:

Review of the "Investigations of the Properties of Positive Ternary Quadratic Forms by Ludwig August Seeber, Doctor of Philosophy, Professor at the University of Freiburg. 1831. 248 pp. in 4."

[Reviewed by C. F. Gauss?]

translated from the German by Sky Shields

[. . .]

We would like now to set down something further regarding the positive binary and ternary quadratic forms which lies outside of the domain of higher arithmetic: it is unnecessary to deal specially with negative forms, and ambiguous forms will be entirely excluded from this treatment.

The positive binary form $axx + 2bxy + czz$ represents generally the square of the distance of two indeterminate points in a plane, whose coordinates differ by $x\sqrt{a}$, $y\sqrt{c}$ with respect to two axes inclined to one another by an angle whose cosine is $= \frac{b}{\sqrt{ac}}$. In so far as x and y are to also signify *whole* numbers, the form corresponds to a system of points ordered into parallelograms, such that the points lie at the intersections of two systems of parallel lines. The lines of each system are at equal distances from one another; so that the distances of one system, if they are measured parallel to the lines of the second, will be $= \sqrt{a}$, and the distances of the second, measured parallel to the lines of the first, will be $= \sqrt{c}$, and the inclination of the two systems with respect to one another will be that given above. In this way the plane appears completely tiled with equal parallelograms whose endpoints constitute the point system, such that no point can fall within a parallelogram. The determinant taken with a positive sign, $ac - bb$, signifies the area of a unit parallelogram [*Elementar-Parallelograms*]. One and the same system of points can be tiled with parallelograms in infinitely many different ways, and can thus be reduced to just as many different forms: however all of these forms are what is termed equivalent, and the area of a unit parallelogram [*Elementar-Parallelograms*] remains always the same. Two forms, which are not equivalent, of which one however implies the other, correspond to the same system of points, but the first form to the entire system, the second to only a part. Two forms which are what is called improperly equivalent in our terminology, correspond to two equal but inverse systems of points, since [*indem*] the plane is considered as inversely distributed, etc.

In the same way, the positive ternary form $axx + byy + czz + 2a'yz + 2b'xz + 2c'xy$ signifies the square of the distance of two indeterminate points in space, whose coordinates with respect to three axes (1), (2), (3) differ by $x\sqrt{a}$, $y\sqrt{b}$, $z\sqrt{c}$: the cosines of the angles between the axes (2) and (3), (1) and (3), (1) and (2) are here $\frac{a'}{\sqrt{bc}}$, $\frac{b'}{\sqrt{ac}}$, $\frac{c'}{\sqrt{ab}}$ respectively. In so far as x, y, z, are here to represent only whole numbers, the form corresponds to a system of regular parallelepipeds, i.e. to points represented by the intersections of three systems of parallel equidistant planes. The entire space thus appears to be completely tiled by equal parallelepipeds, whose endpoints constitute that system of points; and the square of the volume of a unit parallelepiped [*Elementar-Parallelepiped*] is equal to the determinant of the ternary form taken with a positive sign. Equivalent forms represent one and the same system of points, only drawn to other axes or fundamental planes. In the same way, all of the other main points of the theory of ternary forms here find their geometric meaning, the implication of one form by another, the representation of a determinate number or an indeterminate binary form by a ternary form, the theory [*Lehre*] of adjunct ternary forms, the disappearance of the difference between proper and improper equivalence, the nature [*Wesen*] of the reduced form, etc. However we must limit ourselves to the above hints, especially since the work under consideration, which considers the ternary forms from a purely arithmetic point of view, gave rise to them only indirectly. It will at least be recognized what a rich field is opened for investigation, which is not merely of great theoretical interest, but rather can also be used for a treatment, of all of the relations among crystal forms, which will be as convenient as it is general. It is not the place here to go deeper into this application: however we may not forgo the observation that although originally it was assumed that a, b, c, a', b' c', represent whole numbers, the majority of the theory of ternary forms, and namely that which is requisite for this application, remains applicable independently of that assumption. In fact, HAÜY'S statements [*Angaben*] lead to very simple integral values of the coefficients in the ternary forms which correspond to the relevant ordering of the point system; however the more exact later measurements of [Wollaston], [Malus], [Biot], [Kupfer], et al. contradict this, and make it doubtful whether rational ratios of those coefficients are always [*überall*] natural; in any event, however, if it is not desired to reject the limitation to whole number coefficients in the theory, whole numbers can always be found which will come as close as is desired to the measured reults, since in this theory it is not the absolute values, but rather their ratios with one another which matter.

Finally we will develop the geometric meaning of the theorem of SEEBER developed above. If a parallelepiped is so constituted that none of its twelve edges (which are equal in sets of four) are larger either than one of the twelve diagonals of the faces (which are equal in sets of two) or than one of the four diagonals of the parallelepiped, its volume multiplied by $\sqrt{2}$ is therefore not smaller than the volume of a rectangular parallelepiped formed out of the same edges.