Excerpts from

Anfangsgründe

Part 4

Mechanics

Abraham Gotthelf Kästner.

1766

On the Measurement of Force

198. Should a heavy body ascend from K to L (Fig. 5), it must begin its climb with the [the same] known measure of velocity at K, as that which it would have received in virtue of the fall through LK, because in ascent, gravity takes the same velocity from the body, as it had given it in the fall. And likewise, a body, should it ascend from Q to A (Fig.7), must begin to climb at Q with the velocity that the fall through AQ gave it. Thus the square of the velocities with which both bodies begin to climb, is as the height to which they ascend.

199. Therefore, assume that [since] the force with which both bodies climb is as the height to which it climbs, the body that climbs twice as high has twice as much force, thus the force is in square proportion to the velocities; that, should [they] climb twice as high, they must begin their acsent with four times the velocity, and ascend until all of its velocity, and thereby all force to ascend is lost, four times more force than through the ascent of the single height.

200. Thus the two masses M ; m; with initial velocities C ; c; and a mass $N = M$ is imagined, which begins to climb with velocity c, are related only by 199; the evident law, that with the same velocity, the forces will relate as the masses, is therefore

Force of M : Force of $N = C^2 : c^2$
Force of N : Force of $m = M : m$ thus
Force of M : Force of $m = M * C^2 : m * c^2 = M * C2 : m * c2$

or the forces, as defined, relate as the product of the squares of their velocities with their masses respectively. .....

202. This measurment of force has relinquished a controversy whose history

alone would, as modern mechanics is, comprise a volume, since it divides the greatest of mathematical understanding, and a single force has been measured by the the Cartesian in one way, and by the Leibnizians, another. Since the Leibnizian Measure as the word was defined in 199 is evident, as one sees without going far, that the parties in dispute must have taken different meanings of the word **force**. Various proof, which each has offered for their opinion, also make this clear. The exact same actions of the body are to be calculated, from which to evaluate their forces; If one wanted to construct an experiment, one has to take into account the time in which the action occurs, because the same force in different times results in a different action etc. For brevity, it is disputed whether the force is to be measured $C \cdot m$ or by $C^2 \cdot m$, rather whether $C \cdot m$ or $C^2 \cdot m$ should be called force. Therefore this dispute is to give it the right name from the original Greek [*Grundsprache*], one has been Logomachie; yet it is itself such a debate that has been instructive in mathematics; to be fostered as debates in other parts of erudition. Since the manifold investigations, solutions to problems and the like which it has couraged, has yet advanced our knowledge. Therefore, I name some of the hither undertaken pertinent writings, that one will perhaps have the easiest amongst them in Germany, and be able to instruct from it.

203. Johann Bernoullis often cited disc. *Sur le mouvement* *(157. XIII) pertains exactly to this, where the Leibnizian measure of force has been vindicated through various causes. Various essays concerning this controversy are found in the first volume of the **Journal of the Russsian Academy of Sciences** that at that time was fashion, such as: Bilfinger[1] **de Virib. Corp. Moto infitis & illarum mensura** p.43. Wolfs **Principia Dynamica** p.217, both in favor of Leibniz. Wolfs **Principiis Dynamicis** countered Jac. Jurin's[2] *Principia Dynamica*, **Phil Transact**. N.479; art. 4; also Trans. N.476; art 14; communicated an investigation of the measure of force of moving bodies, and proposed a crucial essay in this controversy. Wolf has also treated the measure of force in his *Cosmologia Genereali* Sect. II. C. 4. Johann Bernoulli *de vera notione virium viuar, earunque vsu in dynamicis* Act. Er. Pips. 1735; May p.120-Op. T. III. n. 145. A fairly exhaustive compilation of which, what can be said concerning this, is *Dialago di Vincenzio Riccati della Compagnia di Gesu; dove ne' congressi di piu' giornate, delle forze vive e dell' azioni delle forze morte si tien disforso.* Bologna 1749; 419 Quarter pages. The work is written as a dialogue, a [Vortrag-dissertation, recitation] the Italian still loves very much, and that applies here at least well. *Della Forza de' corpi che chiamano viva libri tre, del Sigr. Francesco Maria Zanotti &c.* **Bol.** 1752; 311. quarter pages, contains many criticisms, especially in III B., against what Riccaati had said for the measure of vis viva. Herr Zanotti has tried, as he himself demonstrates, to define the question merely through metaphysical investigations (col sos discorso metafisico), without what is further assumed from geometry and mechanics as the most known and most common doctrines. *Felicis Anton. Balssi opusc. de viriv. viu.* [das zu Bognon] 1754. 56 quater pages, also contains remarks about Riccati. Sharp objections against the proof of the Leibnizian vis viva are found

in Hrn. v. Mairon's *dissertation fur l' estimation & la mesure des forces motrices des corps* **Nouv.Ed.Par.** 1741; to which *Lettre di Mr. le Mairan a Madame *** fur la question des forces vives* pertains, in which he responds to what the Marquise v. Chatelet in his **Institutions physicques** had opposed of the Leibnizian measure of force. This, along with Fr. Mauquise's response is translated to German by Frau Prof. Gottschedinn , Leips. published in 1742. Various controversial writings on the measure of force in Leipzig are exchanged between the still living Hrn. Pr. Heinsius and the late M. Friedr. Wilh. Stübner. Therein pertains the 1733 first defended disputation under the late Hausen, *de viribus mortricibus*. Stübner's disputation *contra virium mensuram Cartesianam, pro Leibnitiana*; 1733; as well as his *demonstratio verae mensurae virium mortricium viuar. Heinus' Animadersiunes*; Stübner's *amica sepsonsio ad animaduersiones heinsii* 1734; Heins. *Notiones & discrimen virium viuar & mortuar.* 1735. The current Hr. Pr. Arnold has given a short concept of the history of this dispute, in 1754 in *2.disputations de viribus viribus viuis earumque mensura*, in which he defended, first under Hrn. Pr. v. Windheim, the other himself as president. Hausen had published **Programma ad solemnia pronotionis magistror** reaction/counteraction (Gegenwirkung) in a Leipsig 1741 published , which another *ad memoriuam Geyerianam...celbrandam* followed in the same year, where he has stated the known[*Gewisse*] in this dispute very well. I will report the the substance of his thoughts, the unite it with the present system [*Lehrbegriff*] as best as I can, since according to my understanding, nothing better can be written on this subject in such brevity.

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On the conservation of the vis viva

205. In clause(164) this was expressed thus: with elastic bodies, the sum of the living force remains the same before and after impact. This has been called the **Conservation of Vis Viva**, and has been assumed to be a general law of nature. In 114; $Mx^2 + mx^2 = MC^2 + mc^2$ was set, giving $C = c$, thus a condition, under which no impact occurs, and just the same assumption as in 110 is used, gives $c = -C$ just as previously in (111). Therefore, the conservation of living force does not occur with absolutely hard bodies. However even [if] that were a factor (192), such bodies are banned out of the real world.

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On the Theory of Least Action

209. Maupertuis made known a doctrine with this name, which can be used in investigations in mechanics. Since this doctrine has caused much civil commotion, it is necessary here to give at least some account of the matter.

210. **Definition**. The **Quantity of Action**, as called by Mr. M, is the product of mass of a body with its velocity and with the space that it traverses. If a body in one position were brought to another, then the greater the action is, the greater the mass is, the faster the velocity and the longer the space thru which it goes.

211. **Principle of Least Action**. If a change in nature comes to pass, the least possible action is required for this change.

212. **Problem**. The laws of motion of hard bodies derive entirely out of the theory of least action.

Solution: two hard bodies A, B follow each other with the velocities a, b, the former with the greater, so that therefore B will arrive at it [the former] and will be pushed. After the collision, as is easy to see, both have the same velocity, which may be called x. Since in a given time, with uniform motion, the space traversed remains proportional to the velocity, just as it can be said, that the two masses have traversed the spaces a, b in a known time before the collision, and each traverse the space x in the same time after the collision. Thus, the change, which has come to pass is, on a mathematical plane, just as if during the time in which A moves with its velocity a through the space a, this mass, had moved backwards, so that this plane had moved backwards with the velocity $a - x$ through the space $a - x$; and as if B moved through the space b with the velocity b during that time, and a similar plane had moved fowards with the velocity $x-b$ through the space $x-b$. Moreover, of these mathematical surfaces, take the first velocity, which A had before the collision, to be just as much as A lost in the collision and set the other initial velocity to be as much as B gained in the collision. Since now the motion of the surface burdened with bodies is the same, the body may rest or move, then the magnitudes of the generated actions are $A(a-x)^2$; $B(x-b)^2$, whose sum should be simplified; thus their differential set $= 0$: $-2aA+2Ax+2Bx-2Bb = 0$ from which follows $x = \frac{Aa+Bb}{A+B}$, just as in (114.)

213. **Problem**: The laws of motion of elastic bodies are derived from the principle of least action.

Solution: Before a collision, the mass A traverses a space a with the velocity a, after the collision, the space α, with the velocity α; B, b, β signify the same for the other mass as will hold for A. If one repeats the previous consideration (212), then eventually the generated action $A \cdot (a-\alpha)^2$; $B \cdot (\beta - b)^2$ whose simplified sum, will be found to give $-2Aad\alpha + 2Ad\alpha + 2\beta d\beta + 2Bbd\beta = 0$. The collision of the elastic body can now be imagined, as if it were merely hard, and a feather between them compressed between them and stretched out in the collision, however, the feather will be pressed together with the velocity, with which the bodies approach each other, thus with their relative velocities, and consequently with just the force again, as with elastic bodies, the relative velocity before and after the collision is the same, thus $\beta - \alpha = a - b$ and $\beta = a - b + \alpha$, and $d\beta = d\alpha$; this value of $d\beta$ in which is composed of only established equations, gives initially $-2Aa + -2Aa + 2Bb - 2Bb = 0$, and now replace β with its value, giving $-2Aa + 2Aa + 2Ba - 2Bb + 2Ba = 0$ and from that $\alpha = \frac{(A-B)a+2Bb}{A+B}$ and $a-b+a$ or $\beta = \frac{(B-A)b+2Aa}{A+B}$ just as in (135).

214. The previous (210..213); with only some abridgement and explainations, is translated from Mr. Maupertiuss Essay de Cosmologie, which is in particular published in 1750 in the edition of his collected works, and [you will] find the same in book one of the Lyon 1756 published in a four octavo1 part series Oevres de Mr. Maupertuis. Christlob Mylius has translated it into german; Versuch einer Cosmologie2 from Mr. Maupertuis Berlin 1751. 8. The same thing is already in the *Mem. de lAc.Roy. de Pr.* 1746. p.268. u.f. Since, according to this law, as much as possible will be accomplished with as little as possible, it is also called the Law of Parsimony.*

215. I only intended here to give a concept of Herrn von Maupertuiss law. He himself shows yet another application of it, and believes to have therein a fertile axiom of mechanics, which would itself be prefered to conservation of living force (vis viva). Samuel Knig critcizes him, which had been published in Acta. Erud. March 1751. This has provoked a contention, whose circumstantial history, I, out of respect for science, would not like to publish, However, I am fairly completely provided with the thereto relevent informantion and also may posses much [that is] not very well known.

Fortunately, a greater part of this controversy did not even pertain to mechanics, and that has indeed given an opportunity, that people who would not have otherwise bothered themselves around a mathematical question have concerned themselves with this conflict. Mr. Knig quoted a part of a Leibniz letter, which Herr. Maupertuis took to mean that Herr Knig wanted to accuse him of taking Leibnizs discovery for his own, and on this account, challenged Herr. Knig to produce the original letter, and when he couldnt, the Royal Academy of Prussia occasioned a judgement passed on Mr. König, which came out under the title *Jugement de lAc. Roy. Des Sc. & fur une lettre predendue de Mr. De Leibniz to Berlin* 1752.†

König published in opposition, a complaint under the title: *Appel au public du jugement de lAc. Roy.de Berlin &c.* With this occasion Voltaire fabricated against Maupertuis the *histoire du docteur Akakia*, the *Sance memorable* and other comic works. It is strange that Voltaire according to the character(see box), which would otherwise be attributed to him, had again avowed the Law of Parsimony. Under the title: *Vollständig Sammlung aller Streitschriften, die neulich über das vorbliche Gesetz der Natur von der kleinsten Kraft in den*

*Sparsamkeit

†Judgement of the Royal Acadamy of Sciences and a letter purportedly from Mr. Leibniz

Wirkungen der Körper enstanden sind[‡] Leipzig 1753, a collection of these writings, emerged translated into German, which actually had nothing to do with law of least action, but rather, as I have already mentioned, took this opportunity to occupy themselves with another conflict, largely with personal attacks/personality ridden [Persönlichkeit] and insults. Informative in the main points are Mr. Eulers *diff.de principio minimae actionis* Berl. 1753 and in French in the Mem. De l;Ac. De Pr. 1751 p.199; also located on p.169 of *Harmonie entre les principes generaux de repos & de mouvem. de Mr. de Maup.* and p.246 of *essay dune demonstration metaphysique du pr. gen. de l equil.* In the year 1752, p.29 answer of Mr. Maupertuis against Mr. dArch. Ant. Brugmans proeve over de waare grondwetten der beweging en fust [Leid] 1753 which is against Herrn. Maupertuis. Among others, Herrn. Eulers, *Additamenta to the Methodo inueniendi curuas maximi minimiue proprietate baudentes* (gens 1744) pertains to a similar application of the maximum-minimum principle of mechanics; where he first showed how to determine the curvature of elastic plates (curvas elastica) from the consideration of maximum and minimum, then [second] what the minima or maxima will be for a function of the path of a thrown body [i.e. projectile], with no resistance.

[‡] Complete collection of all letters pertaining to the dispute, which recently arose, over the supposed law of nature of the minimum force in the action of bodies.