# Geometrical Writings of Nicolas of Cusa

Originally published in 1796 in Volume I of *Kästner's History*

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Translated from the German by Michael Kirsch*

1. I own a compilation, on the cover of which is written: *Diverse treatises by Nicolaus of Cusa, which extend over pages.* On the other side of this page is a Prohemium.

The beginning of this introduction reads: *In this volume certain treatises and books of the highest contemplation and knowledge are contained: in the clear memory of most excellent and learned individual Nicolaus of Cusa , most Holy Roman Church, Cardinal-Presbyter of St. Peter in Chains: published among many others ....*

The first letter I is missing, in its place is so much space, that it would have reached until the row beneath, where the section which I transcribed ends.

Similarly all the initial letters are missing throughout. That it is all in Gothic script, it is unnecessary for me to remind any expert that this is a known sign that this print belongs among the oldest.

At the end of the mentioned side is an index of the works contained in the compilation. I place it here in its entirety, even though it does not all pertain to mathematics. Each title has its own line but I separate them with —.

*De visione dei —De pace fidei —Reparatio kalendarii — De mathematicis complementis— Cribratio alchoran libri tres— De venatione sapientiae— De ludo globi libri duo— Compendium— Trialogus de posesst— Contra bohemos— De mathematica perectione —De berillo— De dato patris luminum— De querendo deum— Dyalogus de apice theorie—*

2. The format is a short folio, the pages are not numbered at the bottom as is usual, but they are marked with letters, one letter for every six pages, the first the letter is a, b,... then A; my copy goes until the 4th page of letter C; upon whose first side begins with: treatise *On Beryllus expounded.*

It is thus not complete; however, it is in a very fine binding, and following tradition is decorated with engraved figures, well preserved. Bound by: *the Book of Rural Profits (Ruralium Commodorum) by Pietro di Crescenzi at whose bottom reads: this industrious style characterizes the present Book of Rural Profits by Pietro di Crescenzi as a whole printed for the service of the Omnipotent God in the house of John of Westphalia. Nourishing and flourishing at the University of Louaniensi.* No dates, Gothic script. Here the initial letters are inscribed with thick red ink. Also this book has no particular title, but rather begins with: *In the Holy and Indivisible name of the Trinity. Amen* On an empty page is written; *Pietro di Crescenzi — Diverse treatises of Nicolaus of Cusa — Note on the Treatise on the Koran of Mohammet.* Thus this treatise is noteworthy for the old owner, which admittedly, it is not for me.

Also with Petri de Crescentiis book, the date of the printing is not denoted. The general time period can be determined from the name of the printer. It certainly does not need proof that the printing of both books falls in the 15th Century year, only on account of Cusas works do I include the verification, that therein numeral 7 throughout is expressed as was customary toward the end of this century. But 4 is written as it is presently.

3. Allow me to cite something from the first part of the compilation of the Cardinals works, that will be able to be drawn out for mathematics, in so far has the art of geometrical perspective and optics as a basis. *The book, The Vision of God, p. 402* is addressed to *ad abbatem et fraters in Tegernsee.* The preamble gives, for an allegory, a picture that views every face wherever one stands. The Cardinal recounts a few examples [of these images], where they are located, and sends: *a painting: containing the figure of an omnivoyant individual,which I call the "icon of God.".*. Which if they hang it on a wall, and stand in front of it, then the face would look at to everyone, regardless of where they were standing, and if someone walks around in front of it *he will experience that the immobile face is moved toward the east such that it is moved simultaneously toward the west ... and that it observes one motion in such a way that it observes all motions simultaneously. And while he considers in what manner this sight deserts no one, he sees how diligently it is concerned for each one as if it is concerned only with him who experiences being seen by it and not for anyone else.*

I find the Cardinals prayerful meditation of the likeness, theoretically truer, and practically more heart lifting, than what has become stated in the philosophy of our time: God reigns over the whole, without troubling over the individual parts. The rest of the Cardinals thoughts, in which indeed there is much rightness and goodness, dont belong to the present purpose.

4. I can now arrive at my actual intention. *On Mathematical Complements*[1] *To the Most Blessed Father, Nicolaus V. Nicolaus, Cardinal of St. Peter in Chains.*

*Great is the power of the pontifical office which you hold, most blessed Father Nicolaus V: all who consider his powers with attention, equate it to a certain extent to the strongest power, that is there, to transfer the circle into the square and the square into the circle....*

*Recently you have transmitted to me the geometrical writings of the great Archimedes, after they were translated from the Greek, as you received them, through your efforts into Latin. They have appeared so admirable to me, that I had to devote myself to them with all my commitment, and thus has it occurred that as the result of my own research and work I have attached a complement, which I permit myself to dedicate to your Holiness....*

5. Archimedes had measured the circumference through a straight line,

[1] This work has never been translated into english

attempted by means of the spiral, but the velocity of [one the one side] the point, which on the radius moves away from the center, and [on the other] the point, which moves at the end of the radius of the circle, are in proportion as the radius and circle, and this very proportion was sought.

6. The Cardinal begins with an examination of regular polygons. The perpendicular from the center of such a polygon to its side, he calls *prima linea*, and the straight line from the center to the vertex of the angle of the polgon, *secunda linea.* This [latter] is the radius of the circle, which can be described around the polygon.

Then he pictures a series of such polygons, all having the same perimeter as the sides grow in number. The first and second lines differ less, the greater number of sides the polygon has. Thus as the number of sides grows larger, so much closer does the polygon become to a circle, which would have the same circumference. About this polygon, *polygonias issoperimetras*, he undertakes an investigation, [and] gives theorems, for the relationship of the area of such a polygon to the circle, and presents the following problem: Given a straight line, discover the radius of a circle, whose circumference is as long as this straight line. In his proof he uses nothing more than the first and second lines of the isoperimetrical triangle and square.

7. If I have correctly understood his discourse and accurately calculated, then he gives for the circumference = c, the radius = c x 0.102384. Thereby the proportion of the diameter to the circumference would come to 1: 4.8835.

8. He then also inversely transforms the circumference of a given circle into a straight line. The method is theoretically correct and ingenious: from the vertex of a right angle one applies straight lines to both sides, which are in the ratio as I: p, and the hypotenuse is drawn, which is however made longer than between the end points of the sides. This figure becomes constructed from brass or wood( in ere aut lingo). If a circle is now given, then the acute angle [which is opposite the line pie]is layed in circumference of the circle and the line = 1, along the diameter, draw then through the circles center, parallel with the line p until at the hypotenuse. This parallel is the semi circles half circumference.

When the radius of the circle is longer than the line named I , thus the parallel hits the extended hypotenuse.

As the proportion which I call p : I , the Cardinal uses, as is easy to consider: the half of the straight line, which he had assumed, and the radius of the circle, which he had located.

9. The transformation of a square into a circle, among other things. To find the sine and chord, for 1,2,3,4. degrees which no one yet knew.

Between the half of the straight line, which he assumed for the length of the circumference, and to the radius which he found for it, he takes the geometric mean proportional line which is the side of a square having equal area to the circle.

Correct, except that his construction does not correctly give him the ratio of the radius to the circumference.

Then he produces from the apex of a right angle on both sides, the radius and half the side of the square, and draws the hypotenuse, so again he gets an angle to which he shapes from copper or wood, and by means of it finds the square equal to every circle and the circle equal to every square.

Here a circle had been drawn with the square that should be equal to it, whose side clearly will cut the circle, but intersect outside of it. This squares lower side is extended, and the extension is tangent to an equal circle, both circles centers lie above the extended line. From the point of contact a curved line goes upwards, then again downwards, through the center of the square, until about the middle of the squares side which was extended.

I dont find this curved line mentioned in the text. It could occur to someone [that] the circle, [of] which the extended side is tangent [to] at the bottom, should roll along the straight line, and its point which is initially the lowest describe a cycloid: therefore, the straight line, over which the circle turns, must also be tangent with the end of the cycloid, like at the beginning, and the straight line of the figure is tangent to only one of the two circles.

Also I find the rotation of the circle nowhere mentioned here by Cusa, which can so easily occur to one, who seeks the quadrature of the circle,: perhaps he didnt think of it, because he here did not intend, to square a given circle, but rather the inverse, to transform a straight line into the circumference of a circle.

10. Then the cardinal said: *After that which I have previously treated, one can now also attempt what was until today unknown in geometry, namely a final theory of curves and chords (de sinibus et chordis). No one could ever indicate the chord of a curve of one, two, four degrees and so forth; now one can find it. It is certain: in order to produce the radius of an isoperimetric circle, each regular polygon adds a fixed fraction of the difference between its second and first line to its first line. Moreover: The excess, by which the first line of any arbitrary polygon exceeds the triangular first line, the excess by which the triangular second exceeds the second of the other polygon always and in all polygons preserves the same relationship. From this the general theory of curves and chords is elicited; without this theory geometry remained incomplete up to now. But you will find how one can arrive at the practical implementation in approximation numbers in the following. It is impossible in whole numbers, because the square root of 2 (medietas duplae-literally, the mean of two) cannot be expressed in numbers, for the quantity, which this relationship has, is neither even nor odd.*

*The radius of the circle circumscribed by the triangle is therefore 14; then the radius of the associated inscribed circle is 7 [I have mentioned how this number is expressed (2)], the square thereof is 49 and the square of half of the side of the triangle is three times as much, namely 147, the square of the radius of the circle is four times as much, namely 296 [this is what it says in the text, but it should be 196]. Half of the side of the tetragon is now the root of nine sixteenths of the square over half of the side of the triangle, that is, the square root of 82 11/16 [He means 9/16. 147 = the square root of 82 11/16]. That is also the radius of the circle inscribed in the square. The radius of the circle circumscribed in the square is the root of the doubled number, that is, the square root of 165 6/16 [2. (the square root of 82 11/16) = the square root of 165 6/16]. If the square root of 49 is now subtracted from the square root of 82*

*11/16, then this difference denotes the excess of the radius of the circle inscribed in the square over that in the triangle and amounts to something more than 2; if one subtracts the square root of 165 6/16 from the square root of 196, then this difference amounts to something more than 1. Thus, you have the differences between the prime on the one side and the second on the other, and everything further can be pursued from the relationship of these differences. Namely, if you subtract this difference from the sagitta of the side of the triangle, that is from 7, the sagitta of the square remains; if you now divide 7 according to the relationship of the difference given above and add the larger section to the radius of the circle inscribed in the triangle, you have the radius of the isoperimetric circle.*

*In this way you can also provide the square of any arbitrary polygonal side from the square of the side of the triangle and of the side of the square; from this and from the relationship of the differences one comes to the sagitta and to the radius of the inscribed circle, and thus one knows the curvature of the chord, and this is the final completion of the geometrical theory, to which the ancients, as far as I have read, had not advanced. Now the theory of the geometrical transformations is also completed, which earlier I have adequately described more briefly, as far as it concerns the quadrature of the circle.*

I have placed this passage here, because in it *sinus* and *sagitta* became named. At the beginning of it I could anticipate, that it would be proven how its chord or sine of a degree etc, would be given, but at the end that was far from the case. Nevertheless, such chords had already been given by Ptolemy, and accordingly it also was given by the Arab in the Almagest, which could not have been unknown to the Cardinal. I thus do not see, how he promised to accomplish something out that the ancients did not accomplish, for in any case he only wanted to yield approximately that which was desired.

Admittedly the accomplishment would have been rather difficult due to the very incomplete state of his arithmetic, and thus did not achieve complete accuracy.

From $82 + 11/16 = 8.6875$ gives the logarithm $= 1.9173874$ , which halved $= 0.95869378$ which belongs to 9.0927. 7 subtracted from this, leaves 2.0927, which the Cardinal called a little more than 2. With such an entirely superficial estimation of figures he could not advance further, even if the theory were accurate, from which he derived it. He thus flattered himself too much, as to Pope, when he said about its construction with angles, etc. *To whomever wants to exert his genius, it becomes clearly accessible. Hence this invention rightfully obtains the name Complement, and deserved to become generally well-known through your wonderful power, Most Blessed Father, which astonishes all Catholics, so much that they name you after the name of admiration, father of fathers.*

About lines and figures, which arise, if a straight line moves or rotates, while a point moves along it. At the conclusion: to find the sides of polygons which are equal to the circle.

11. For the times in which the cardinal lived it indicates an extraordinary spirit and passion to perceive what was to be discovered, and to attempt for it,

even if that attempt was not sufficient.

The comparison between his first and second lines and sides of the isoperimetrical polygon can presently be given through the formulas of analytical trigonometry; he could hardly represent it exactly for every individual polygon solely through common arithmetic . I surmise, he had even determined the first and second lines for the triangle and square solely through diagrams, because he conveyed everything onto diagrams; and when he wants to illustrate its composition with numbers, he is absolutely not concerned to be accurate or to come close to being exact, but only to use it as a example.

Among those, who have occupied themselves with cyclometry, I dont know any one else, who took a given straight line equal to the circumference, and sought the radius which belongs to it. He was led to it by the isoperimetric polygons.

12. The content the book *de venatione sapientiae*, is shown by its title. Among the means which the Intellect employs to hunt wisdom, Chapter V calls to notice also: *Quomodo exemplo geometric perficit.* The content is, that the geometrical ideas in the mind are never perfectly represented through their sensual images; one seeks only, that the images of the ideas are precise as possible and is required of the image.

13. In the book *de ludo globi* perhaps Mathematics could also be expected. It is a dialogue, which is presented: *Nicolaus, Cardinal of St. Peter in Chains, and John, Duke of Bavaria. The The duke begins, The Duke begins: Since I have seen that you have withdrawn to your seat, perhaps tired by the game of spheres, I would like to confer with you about this game, if it is agreeable to you.* The duke observes, that there must indeed be something more to be considered about this game because it so pleasing to men, and the Cardinal acknowledged this, for some sciences also have their own game: *Arithmetic has its number games, music its monochord, nor does the game of chess lack a moral mystery.*

The Cardinal observes further: no brute beast moves a ball to its goal. *Therefore you see that the works of man originate from a power which surpasses that of other animals of this world.*

The ball, which was used in this game, must have had a certain metaphor. *I do not think you are ignorant of why the ball, through the art of the turner, assumes a hemispherical shape that is somewhat concave. For it if did not have such a shape, the ball would not make the motion that you see: helical/vertiginous, that is spiral or involuted. For part of the ball, which is a perfect circle would be moved in a straight line, unless its heavier and corpulent part retarded that motion and drew the ball centrally back to itself. Based on this diversity the shape is capable of a motion, which is neither entirely straight nor entirely curved, as it is in the circumference of the circle, which is equidistant from its center. From this you will first observe the reason for the shape of the ball, in which you will see the convex surface of the larger half sphere and the concave surface of the smaller half sphere. And the body of the sphere is contained between them. You will then see that the ball can be varied in infinite ways according to the various conditions of the described surfaces and can always be adapted to one and the other motion.*

14. The cardinal gives the following report, not far from the end of this book: *However, it was my intention to apply this recently invented game, which everyone easily understands and gladly plays, because of the changing and never certain course of the ball, in a manner useful to our purpose. I have made a mark where we stand when throwing the ball. And a circle in the center of the level ground. In its center, enclosed in the circle, is the seat of the king, whose kingdom is the kingdom of life. And in this circle are nine others. However, the law of the game is to make the ball stop moving inside the circle. And the closer it comes to the center, the more it acquires, corresponding to the number of the circle, in which it comes to rest. And whoever is the first to attain 34 points, that is the number of the years of Christ, is the winner.*

*This game, I say, signifies the motion of the soul from its kingdom to the kingdom of life, in which is peace and eternal happiness. Jesus Christ, our King and the giver of life, governs in its center. Since he was similar to us, he moved the sphere of his person, so that it came to rest in the middle of life, leaving us the example, so that we would act just as he acted. And our sphere would follow him, even though it would be impossible for another sphere to attain peace in the same center of life where the sphere of Christ rests. Inside the circle there is an infinity of locations and mansions. For each persons locus rests on its own point and atom, which no one else could ever attain. Nor can two spheres be equidistant from the center, but one is always more, the other less so. Therefore it is necessary that all Christians contemplate how some do not have the hope of another life and that they move their sphere in earthly domain. Others have the hope of happiness, but they attempt to achieve that life by their own powers and laws without Christ. And they make their sphere run to higher things by following the powers of their own genius and precepts of their own prophets and teachers. And their sphere does not reach the kingdom of life. There is a third group, which embraces the life that Christ, the only begotten son of God preached and walked. They turn to the center where the seat of the king of virtue and of the mediator of god and man is. And following the vestige of Christ, they bring their sphere onto a moderate course. These alone acquire a mansion in the kingdom of life. For only the Son of God, descending from heaven, knew the way of life, which he revealed in word and deed to the believers.*

I thought, the long passage deserved to be distinguished, because besides demonstrating the composition of the game it also demonstrates a remarkable theological intention. Maybe a game with a ball, which must be left to rest inside a certain boundary, was common, and the Cardinal adjusted it for his purpose. In any event he gives himself as the inventor somewhat before the quoted passage. Freedom, he says, is mans superiority over the beasts, as beasts of one species all act the same concerning prospecting its food, building nests etc, always one as the other; while every man acts according to his own wisdom. *When I invented this game, I thought, I considered, and determined, that which no one else thought, considered or determined.*

Indeed, the structure of what he calls a ball is also peculiar, of which could well be desired a more exact description but an intelligible description, a useful illustration, was not required in that time. If such a thing did exist, the shape and path that it would take by a given impulse, could keep an Euler busy.

15. At the time all that was known, was that with every shot of the ball it would take a different path, because each time it would in a different manner be held, let go from the hand, laid onto the ground, collide: It is not possible to do something the same way twice, for it implies a contradiction that there be two things that are equal in all respects without any difference at all. How can many things be many without a difference? And even if the more experienced player always tries to conduct himself in the same way, this is nevertheless not precisely possible, although the difference is not always perceived.

Here one has Leibniz *principium indiscernibilium (Principle of the Indiscernible).*

16. The visible rounding could not be perfect, *the outermost edge of the roundness is terminated in an indivisible point that remains entirely invisible to our eyes. For nothing can be seen by us unless it is divisible and has size.* The significance is well only this: whether the spherical curvature were geometrically perfect, or depart insensibly from it, can not be perceived with the senses. Then the dialogue passes onto the roundness of the universe, motion, and philosophy, morality, even theological teachings. Even if there was place for it here, it would be too much effort to clearly represent it, as even the Verses at the close of this book say in praise of the same. They begins thus: *What genius you desire at present in our little book*

*First repeat the holy reason three times, four times,*
*And more than once: understanding as soon as you survey the heights*
*At the top: And titles are reduced to empty reason.*

17. There follows yet a second book *de ludo globi,* here the people in discussion, the young man Albert, The Duke of Bavaria and Nicolas of Cusa. Albert has seen that his relative Johann read the book de ludo globi, and comes to the Cardinal in request of further explanation. *It didnt seem to me, he says, that you explained the mystical meaning of the circles of the region of life.* Theorems appear here as before in the first book, which are sometimes explained with geometrical likenesses, for example, through circles and rotation of the circle.

18. The book *de mathematica perfectione* This work has also never been translated into englishis: dedicated *to the Most Reverend Father in Christ, the Lord Antonius, of the Holy Roman Church Cardinal-Presbyter of St. Chrysongonus, by Nicolaus, Cardinal of St. Peter in Chains.* Then he says: *However, that mathematical insights lead us to the entirely absolutely divine and eternal, your paternal Grace knows better than I, according to the extent of your high erudition, You who are the summit of theologians.*

19. He begins with: whether the smallest chord, of which there cannot be a smaller could be given, then it would have no sagitta, and were not small as its arc; reason concieves that, although it knows, that neither the chord or the arc can be comes so small, that it cannot become smaller, *since the continuum is infinitely divisible.*

20. He now imagines a right angled triangle, whose hypotenuse *linea prima*, is the radius of a circle, whose arc measures the angle opposite the smallest side (its *linea secuda*) Thus, this angle can be no larger than 45 degrees.. He calls

the third side *linea tertia*, the arc *simiarcus*, the second line *semicorda*.. That is to say the half of the chord, of the arc, of which the given would be half.

Then he says the following: the named half Arc is to the half chord, as triple of the first line, is to the sum of the third line plus the twice the first.

21. I call Cusas first line or the hypotenuse r; the triangles angle $a$; thus the length of arc described with $r = ra$; the second line = *rsina*, the third line = *rcosa*; and the Cardinal says: $ra/rsina = 3r/2(r+r)cosa$; thus $a = 3\ sine\ a/2 + cos\ a = 3\ tang.a/2\ sec\ a + 1$

The composition is only true, when the Arc and Sine vanish together. Thus for small arcs truth is close, for the greater is always removed more from it, and furthest, when the angle = 45 degrees; since I find by means of the logarithm 3/2 Sec 45 +1 = .78361; therefore the arc for 45 degrees is = 0.78539; thus the maximum defect known in those times, when only the Archimedian proportion of the diameter to the circumference was known, which was limited to hundredths of the diameter.

22. The cardinal could not have proved the theorem. His justification of it is fairly obscure, and to explain it would only worth the trouble if it could contain the truth. Only so much of it deserves to be brought forward that would give an idea as to how he might have come upon the theorem. He assumes one and the same straight line, added to the first and third side of every triangle, and gives a sum, which is proportionate as the arc to the second side. Then it can to be solved from his discourse, that this line would be the double of the first side, in the biggest triangle, whose angle opposite the second side is = 45 degrees. Where he knows that, he does not mention; maybe he has discovered it through trials, and thereby assumed this magnitude of the quadrant as well as he knew; his operation could not have been very precise, otherwise he would have perceived that it did not concur with his assumption.

Then he said, what occurs in this maximum triangle, occurs also in the minimum, if the same thing could happen, as when the third link would not surpass the second; thus it occurs also with all the triangles in between. *And that is the root of this teaching; from it follows: If I find the line, which is to be added to the right-angled triangle with bc as half-chord of the quadrant and in the hexagon with bc as half-chord, then the sums found are in the same ratio as the arcs, i.e. they are as 3 to 2. It is clear that I have therewith found the line, which is to be added in all cases and there is no doubt about it.*

Unquestioned at any rate is, that the Cardinal expressed himself very incomprehensibly.

23. A number of applications of this theorem to the measure of the circle and the sphere. The close of the book is: *In a similar manner, you yourself may derive the relationship with regard to the minimum in other curved surfaces. What can be known in mathematics in a human manner, from my point of view, can be found in this manner.*

That sounds like an introduction of the analysis of the infinitesimal calculus. Thus one could say something to the cardinal which he had not considered. In fact, he contemplated evanescent magnitudes, only he did not know how this conception would be used.

24. In the book *de berillo* there are frequently straight lines and angles which are meant to explain philosophical, theological teachings. *Beryllus is a lucid, white, and transparent stone. It is given at the same time a concave and convex form, and looking through it, one attains to things with intellectual eyes which were previously invisible. This book I meant to accomplish the same for the intellect.*

25. More efforts of the Cardinal about the quadtrature of the circle of which Regiomantus spoke, find themselves drawn out in the book *de triangulos, where I also discussed it.*

**All Italics were originally in Latin, which Kstner quoted from his Book. All Latin was Translated By William F. Wertz Jr.*