Excerpts from *Anfangsgründe* Part 1 Section 1 *Geometry and Trigonometry*

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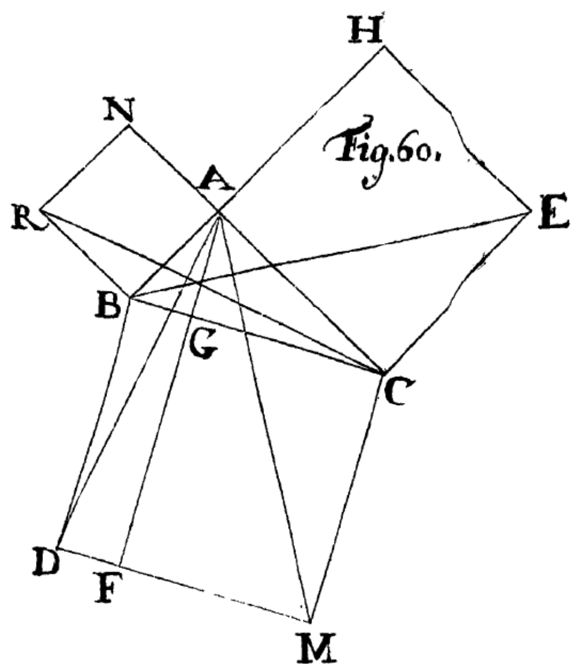
Theorem 15.

Lemma. In a right triangle BAC (*Fig. 60*), the square $BCMD$ of the hypotenuse BC is as great as the sum of the squares $ABRN + ACEH$ of the sides AB and AC.

Proof. Drawing a perpendicular to the hypotenuse from the right angle divides its square into two rectangles $BGFD$ and $CGFM$. Each of these is equal to the square which lies on its side.

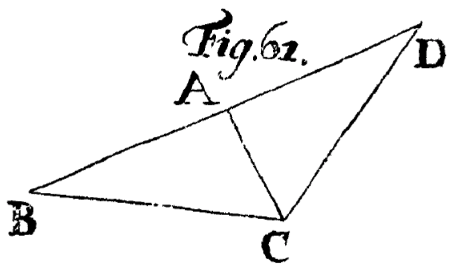
For, drawing AD, CR, then $\Delta BAD = \frac{1}{2}BGFD$ and $\Delta RBC = \frac{1}{2}RNAB$ (*Theorem 14. 3 Corollary.*). However, $RB = BA$, $BC = BD$, and $RBC = 90^o + ABC = ABC + 90^o = ABD$. Thus, $\Delta RBC = \Delta BAD$ (*Theorem 2.*) and therefore $RNAB = BGFD$.



In the same way $\Delta CEB = \frac{1}{2}CEHA$; $\Delta CMA = \frac{1}{2}CGFM$; and $CE = CA$; $CB = CM$; $ECB = ACM$; therefore $\Delta CMA = \Delta CEB$ and $CGFM = CEHA$.

Consequently, the sum of the rectangles, or $BCMD$, is equal to the sum of the squares; that is, $BC^2 = AB^2 + AC^2$.

1. Corollary. Conversely, if $BC^2 = AB^2 + AC^2$ (Fig. 61), then $BAC = 90^o$, since, drawing the perpendicular $AD = AB$ from CA, then $CD^2 = CA^2 + AD^2 = CA^2 + BA^2 = CB^2$, thus $CD = CB$, $\Delta CAD = \Delta CAB$ (*Theorem 4.*), and $CAB = CAD = 90^o$.



2. Corollary. To construct a square, which is equal to the sum of two squares, the sides of the squares are brought together at a right angle and an hypotenuse is drawn, which will be the side of the sum. However, in order to find the line whose square is equal to the difference of two squares, from the smaller side, CA, is erected an unlimited perpendicular, AB, and with this intersecting the greater side CB, thus is $CB^2 - CA^2 = AB^2$.

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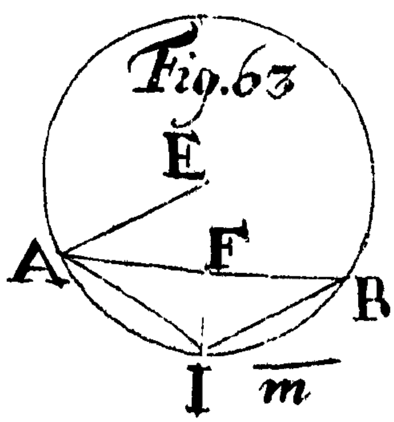
Theorem 43.



Task. Given, in (*Fig. 63*), the radius EA and the side AB of a regular polygon, find the side of the polygon, AI, which has twice as many sides.

Solution. $EF = \sqrt{(EA^2 - AF^2)}$ (*Theorem 15.*) is given, since $AF^2 = \frac{1}{4}AB^2$; thus, if EF is subtracted from $EI = EA$, then FI is given, and consequently $AI = \sqrt{AF^2 + FI^2}$. Let $EA = r$, $AB = f$, then $EF^2 = r^2 - \frac{1}{4}f^2$ and $FI = r - EF$; thus, $AI^2 = \frac{1}{2}f^2 + (r - EF)^2 = \frac{1}{4}f^2 + r^2 - 2r \cdot EF + EF^2 = \frac{1}{4}f^2 + r^2 - 2r \cdot EF + r^2 - \frac{1}{4}f^2$, hence $AI = \sqrt{2r^2 - 2r \cdot EF}$.

Example. If (*Theorem 23. Corollary 6.*) $f = r = 1$, then $EFq = 1 - \frac{1}{4} = \frac{3}{4}$; thus, $EF = \frac{\sqrt{3}}{2}$ (*Arithmetic I. 77.*) and consequently, the side of the dodecagon

$= \sqrt{2-\sqrt{3}}$. Now since $\sqrt{3} = 1.732051$ (*Arithmetic IIII. 27*), then the side of the dodecagon $\sqrt{0.267949} = 0.517$. If still smaller parts of the radius are required, then $\sqrt{3}$ must be extended further to the more exact [value] 0.51764.

1. Corollary. If f is the side of a polygon of n sides, and the side of the polygon of $2n$ sides is to be found from this, then this [value] is substituted into the expression $\sqrt{2r^2 - 2r \cdot EF}$, where $EF = \sqrt{r^2 - \frac{1}{2}f^2}$, in the place of f. Thence, from this can be found the side of the polygon of $4n$ side, and using this in place of f, yields the site of the polygon of $8n$ side, and this can proceed without end.

Thus from the dodecagon is found the 24-gon: from this the 48-gon, and so on.

2. Corollary. The smaller f is, the less EF differs from r. According to this theorem an f can always be found, which is smaller than any given length, and consequently, a polygon for which EF differs from r less than any given difference amounted.

Example. Thus, let $f = 0.517$ for the dodecagon. Then $\frac{1}{2}f = 0.2585$, $\frac{1}{4}f^2 = 0.06682225$, and thus $1 - \frac{1}{4}f^2 = 0.93317775$, which, extracting the root therefrom, gives 0.966 as the perpendicular of the dodecagon. This is, however, somewhat too large, since f was taken too small. More precise is 0.96592.

3. Corollary. The sides (*Corollary I.*) become chords (*Theorem 41.*), which are always approaching their arcs. Thus, if g, h, i, k are the sides of polygons with n, $2n$, $4n$, $8n$ sides, then $2n \cdot h$ will differ from the circumference of the circle less than $n \cdot g$, $4n \cdot i$ less than $2n \cdot h$; $8n \cdot k$ less than $4n \cdot i$, and thus this approximation proceeds without end, such that a polygon can always be found, whose circumference differs from the circumference of the circle by less than any given magnitude (*Theorem 41. Corollary I.*). A polygon of very many sides will thus come very near the circumference of the circle, and one of more sides even nearer, etc. such that a method is now seen, how the circumference of the circle can be expressed by an approximation.

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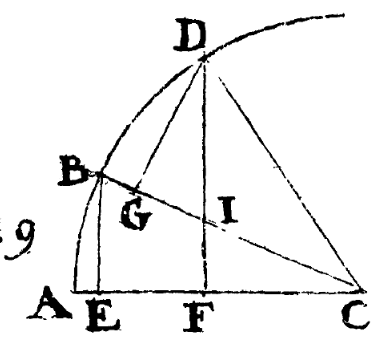
Application of Algebra to Find the General Theorem of Trigonometry

Theorem 19.

Problem. With radius $CA = r$, (*Fig. 9*), the arcs AB, BD, AD, have sines $BE = m$, $DG = d$, $DF = s$, and cosines $CE = \mu$, $CG = \delta$, $CF = \sigma$. One seeks **I**) the sine of the difference d in terms of the sines of the two arcs; **II**) the sine of the sum s in terms of the sines of the two arcs.

Fig. 9



Solution. **I**) $CE : EB = CF : FI$, therefore $FI = \frac{m\sigma}{\mu}$, and $CI = \frac{CF \cdot CB}{CE} = \frac{r\sigma}{\mu}$, $DI = s - IF = \frac{\mu - m\sigma}{\mu}$, $DG = \frac{DI \cdot CF}{CI} = \frac{\mu s - m\sigma}{\mu} \cdot \sigma \frac{\mu}{r\sigma} = \frac{\mu s - m\sigma}{r} = d$.

II) $DI = \frac{CI \cdot DG}{CF} = \frac{rd}{\mu}$, $IG = \frac{DG \cdot FI}{CF} = \frac{DG \cdot EB}{CE} = \frac{md}{\mu}$. Therefore, $CI = \delta - \frac{md}{\mu} = \frac{\delta\mu - md}{\mu}$ and $IF = \frac{CI \cdot BE}{CB} = \frac{\delta m\mu - mmd}{\mu r}$ and $DF = DI + IF = \frac{r^2 d + \delta m\mu - mmd}{\mu r} = \frac{\mu\mu d + \delta m\mu}{\mu r} = \frac{\mu d + \delta m}{r} = s$.

Scholium. To spare oneself the pains in writing, one can set $r = 1$; thence, $d = \mu s - m\sigma$ and $s = \mu d + \delta m$. Since the sine and the radius are lines, then there must be equally as many lines multiplied with one another in every part of the equation, where the amount of those in the dividend is subtracted from the amount of those in the divisor. This is called the equality of the dimensions (*dimensionum*), and if the radius is to be restored to r instead of 1, then it must make all of these dimensions equal.

1. Corollary. Thus, $d^2 = \mu^2 s^2 - 2\mu m s\sigma + m^2\sigma^2$. Subtracting from this $1 = s^2 + \sigma^2$ leaves $m^2 s^2 + 2m\mu s\sigma + \mu^2\sigma^2$ or $\delta = ms + \mu\sigma$.

2. Corollary. Similarly, subtracting s^2 from $1 = d^2 + \delta^2$ yields a square whose root can be $\mu\delta - md$ or $md - \mu\delta$. Each root can represent σ. Taking both angles ACB, BCD to be acute, and their sum to be likewise, which is the case assumed in the figure, then the calculation is arranged. Thus, $\sin BD < \cos AB$ or $d < \mu$; consequently, $1 - \mu^2 < 1 - d^2$ or $m < \delta$: therefore $md < \mu\delta$. Now, since in the assumed case σ should be positive, the expression $\sigma = \mu\delta - md$ must be used. Were other cases to give a negative value for σ, the positive should still be used.

3. Corollary. For the angles ACB, BCD, ACD, let t, y, T be tangents and p, q, P be secants, thus $p : t = 1 : m$ and $p : 1 = 1 : \mu$ (*Definition 4. Corollary 5.*) or $m = \frac{t}{p}$, $\mu = \frac{1}{p}$. Likewise, $d = \frac{y}{q}$, $\delta = \frac{1}{q}$, $s = \frac{T}{P}$, and $\sigma = \frac{1}{P}$. Setting these values in place of the sines and cosines in equation **II** of the solution produces $\frac{T}{P} = \frac{y}{pq} + \frac{t}{pq}$; and similarly will the equation from (*Corollary 1.*) give $\frac{1}{q} = \frac{Tt+1}{1-ty}$, which transforms the just found equation into $Tp^2 = (y+t)(Tt+1)$, wherefrom, after the secants are removed, is obtained $T = \frac{y+t}{1-ty}$. This would be found through a somewhat shorter calculation if instead of [the equation in] (*Corollary 1.*), [that of] (*Corollary 2.*) were used, since the indicated root of the square corresponding to this case must be taken.

4. Corollary. The tangent of the difference is $y = \frac{t-t}{Tt+1}$.

Example. If $AB = 45^o$ and $BD = 30^o$, therefore $m = \frac{\sqrt{2}}{2} = \mu$, $d = \frac{1}{2}$, and $\delta = \frac{\sqrt{3}}{2}$ (*Definition 2. Corollary 1.*). Thus, respecting (*Corollary 2*), $\sigma = \frac{\sqrt{2}}{4} \cdot (\sqrt{3}-1) = \cos 75^o$. Further, $t = 1$ and $y = \frac{1}{\sqrt{3}}$, thus $T = \frac{1+\sqrt{3}}{\sqrt{3}-1} = \tan 75^o$.

Scholium. If the angle is obtuse, then such is indicated by [taking] its cosine and tangents, which are negative, into this formula. If $AB = 45^o$ and $BD = 60^o$, then $t = 1$, $y = \sqrt{3}$, and $\tan 105^o = T = \frac{1+\sqrt{3}}{1-\sqrt{3}} = \tan 75^o$.