Excerpts from *Anfangsgründe* Part 3 Section 1 *Analysis of Finite Magnitudes*

Abraham Gotthelf Kästner.

1760

Solution of Determined Cubic Equations through the so-called Cardan Formula.

699. Task. To find an irrational expression for the roots of the cubic equation $x^3 = ax + b$. Every cubic equation can be reduced to this form (*282*).

Solution. Set $x = \sqrt[3]{A} + \sqrt[3]{B}$, and further suppose, A and B should both be roots of the following quadratic equation, $z^2 = \alpha z - \beta$. In order to determine α and β by means of the given coefficients a, b, I consider $A + B = \alpha$, $AB = \beta$ (*224*), and

$$\begin{aligned} x^3 &= A + 3\sqrt[3]{A^2B} + 3\sqrt[3]{Ab^2} + B \\ &= A + B + 3\sqrt[3]{AB}(\sqrt[3]{A} + \sqrt[3]{B}) \\ &= 3x\sqrt[3]{AB} + (A + B) \end{aligned}$$

From this follow, $\alpha = 3\sqrt[3]{AB} = 3\sqrt[3]{\beta}$, $b = A + B = \alpha$. Consequently, $z^2 = bz - \dfrac{a^3}{27}$. Hence, $z = \frac{1}{2}b \pm \sqrt{\frac{1}{4}b^2 - \dfrac{a^3}{27}}$. Designating the irrational part by c, then $A = \frac{1}{2}b + c$, $B = \frac{1}{2}b - c$, $x = \sqrt[3]{\frac{1}{2}b + c} + \sqrt[3]{\frac{1}{2}b - c}$, instead of which I shall write $m + n$.

700. Corollary. Dividing the cubic equation by $x - m - n$ (*226*), yields the final remainder $-b - a(m+n) + (m+n)^3$. However, it was previously found that $[x^3 =]\ (m+n)^3 = A+B+3\sqrt[3]{AB}(\sqrt[3]{A}+\sqrt[3]{B}) = b+a(m+n)$. Thus, adding $-b - a(m+n)$ cancels out. The quotient then is $x^2 + (m+n)x + (m+n)^2 - a$. Here $3mn = a$, therefore, $(m+n)^2 - a = m^2 + n^2 - 3mn$. Now the roots are $\dfrac{m + n \pm \sqrt{(m+n)^2 - 4(m^2 + n^2 + 3mn)}}{2}$. Therefore, these roots become $x = \dfrac{m + n \pm (m-n)\sqrt{-3}}{2}$.

701. Corollary. If c is possible, then m and n are possible, therefore two roots of the cubic equation (*699*) are impossible.

702. Theorem. Let K be a possible magnitude, and λ an impossible one whose square is possible. For example, $\lambda = L\sqrt{-1}$ where L is possible. Further, P, Q should also designate possible magnitudes. K, L, P, Q may now be rational or irrational magnitudes. Then, **I**) $\sqrt[3]{K+\lambda} = P + Q\sqrt{-1}$; **II**) $\sqrt[3]{K-\lambda} = P - Q\sqrt{-1}$.

Proof. It is immediately evident that the cubic root of the sum of a possible and impossible magnitude is impossible: however, it must also have a possible part, for an impossible magnitude alone cannot produce the possible part K in its cube. Taking the cube of both sides of **I**) and setting the possible equal to the possible and the impossible equal to the impossible (approximately according to the manner concluded in (*235; 233*)), then $K = P^3 - 3PQ^2$ and $\lambda = (3P^2Q - Q^3)\sqrt{-1}$ or $L = 3P^2Q - Q^3$. Consequently, $K - \lambda = P^3 - 3P^2Q\sqrt{-1} - 3PQ^2 + Q^3\sqrt{-1} = (P - Q\sqrt{-1})^3$. Therefore it becomes evident that the equations **I**) and **II**) are real together, or that the second follows from the first. It is only still to be proven, that P, Q are possible magnitudes. This manifests itself in the following form: $(K+\lambda)(K-\lambda) = K^2 - \lambda^2 = K^2 + L^2$, which is certainly a possible magnitude. Therefore its cubic root is as well. Denote this cubic root e. Then from the course of the proof, $(K+\lambda)(K-\lambda) = ((P+Q\sqrt{-1})(P-Q\sqrt{-1}))^3 = (P^2+Q^2)^3$. Therefore $e = P^2 + Q^2$ or $Q^2 = e - P^2$. Substituting this into the equation $K = P^3 - 3PQ^2$ gives **A**) $P^3 - \frac{3e}{4}P = -\frac{1}{4}K$; then, similarly placing $P^2 = e - Q^2$ into $L = 3P^2Q - Q^3$ gives $Q^3 - \frac{3e}{4}Q = -\frac{1}{4}L$, where P and Q each have a certain possible value (*272*). For the proof it is sufficient that it gives a possible value for each of the magnitudes. If they still have impossible [values], it is negligible. The proposition is true in the very sense in which it was maintained, such that the cubic root of a possible magnitude is possible, although this cubic root still always has two impossible values (*240*).

703. Corollary. This consideration just cited [*angeführte*] displays what both of the remaining values of P in **A** are. Namely, from (*240*) as well, $\sqrt[3]{K+\lambda} = \dfrac{(P+Q\sqrt{-1})(-1 \pm \sqrt{-3})}{2} = -\frac{1}{2}P \mp \frac{1}{2}Q\sqrt{3} \pm \frac{1}{2}P\sqrt{-3} - \frac{1}{2}Q\sqrt{-1}$. Thus, the root in **A** signifies the possible part of the cubic root of $K+\lambda$, which is also, therefore, signified by $-\frac{1}{2}P \mp \frac{1}{2}Q\sqrt{3}$.

Example. If $K = 81$, $\lambda = 30\sqrt{-3}$, then $n = 21$ and $P^3 - \frac{63}{4}P = \frac{81}{4}$. Here $P = -3$, $Q = 2\sqrt{3}$ or $(-3 + 2\sqrt{3})^3 = 81 + 30\sqrt{-3}$, which can be affirmed just as easily by trial, if it is also not known, as P and Q are found. Thus, the roots of the equation **A** are -3, $-\frac{3}{2}$, and $\frac{9}{2}$. It becomes apparent, on this account, that Q is determined by a cubic equation.

704. Scholium. Were the shortening of the calculation by the use of the magnitude e not followed, as Clairaut taught in the German translation of his

Algebra 256, then, removing Q (*199*) from both of the equations, which, in the beginning of the proof, were set $K = \ldots$, and $L = \ldots$, thereupon an equation of the 9th degree results for P, which can be reduced to a cubic equation. Landen arrives at this in his *Mathematical Lucubrations*, pg. 54.

705. Theorem. If c, (*699*), is impossible, then m and n are also impossible (*702*), and thus the three roots of the cubic equation (*699*) are possible.

Proof. Thereupon, $m = P + Q\sqrt{-1}$, $n = P - Q\sqrt{-1}$ (*702*). Consequently, $m+n = 2P$; $m-n = 2Q\sqrt{-1}$, which, according to (*700*), is multiplied by $\sqrt{-3}$, giving $-2Q \cdot 3$. Therefore the three roots are $2P$ (*699*) and $P \mp 3Q$ (*700*).

706. Corollary. Since c is possible or impossible according to whether $\frac{1}{4}b^2 - \frac{1}{27}a^3$ is positive or negative, then the cubic equation (*699*) has purely possible roots, or two impossible roots, according to whether $\frac{1}{4}b^2 - \frac{1}{27}a^3$ is negative or positive [respectively].

707. Corollary. If the cubic equation has two impossible roots, then, from the procedure of (*699*), $m+n$ yields a possible value for the remaining possible root.

708. Corollary. Were, however, all three of the roots of the cubic equation possible, m and n would become impossible, such that, therefore, the just mentioned procedure would not, as it seems, be able to establish three possible roots. This apparent difficulty shall be lifted at once.

709. Theorem. If the cubic equation has three possible roots, then the procedure of (*699*) produces all three.

Proof. Thereupon, $\sqrt[3]{A} = P + Q\sqrt{-1}$ and also $= (P + Q\sqrt{-1})\frac{-1\pm\sqrt{-3}}{2}$. Similarly [*imgleichen*], $\sqrt[3]{B} = P - Q\sqrt{-1}$ and also $= (P - Q\sqrt{-1})\frac{-1\pm\sqrt{-3}}{2}$ (*240*). Now, should $3\sqrt[3]{AB} = a$ be possible, then, in order to constitute x, whose product is possible, such expressions of the cubic roots must be taken together. Consequently, the three values of x are:

1	$P + Q\sqrt{-1} + P - Q\sqrt{-1} = 2P$
2,3	$(P + Q\sqrt{-1})\dfrac{-1 \pm \sqrt{-3}}{2} + (P - Q\sqrt{-1})\dfrac{-1 \mp \sqrt{-3}}{2} = -P \mp Q\sqrt{3}$

If the upper sign of the impossible root is taken for the second value of x, the lower being used for the third, then $\sqrt[3]{AB} = (PP + QQ)\left(\dfrac{-1-(-3)}{2}\right)$ obtains a possible value. Conversely, this product and its cubic root would become impossible, were $(P + Q\sqrt{-1})\dfrac{-1 \pm \sqrt{-3}}{2} + (P - Q\sqrt{-1})\dfrac{-1 \pm \sqrt{-3}}{2}$ assumed.

710. Theorem. If the cubic equation has two impossible roots, then the procedure in (*699*) produces such, in addition to the possible root (*707*).

Proof. Thereupon, according to (*240*), x also obtains both of the following values in addition to the value $m+n$ consisting of two possible parts:

$$2,3 \;\Big|\; m\frac{-1\pm\sqrt{-3}}{2}+n\frac{-1\mp\sqrt{-3}}{2}=-P\mp Q\sqrt{3}$$

from which becomes $\dfrac{-(m+n)\pm(m-n)\sqrt{-3}}{2}$. Since m and n are not the same, the impossible part does not cancel itself out there.

711. Corollary. If factors [*Wurzelgleichungen*] (*220*) are constructed from the three values of x (*709*), and multiplied, then $x^3-3(P^2+Q^2)x-2(P^3-3PQ^2)=0$ results. However, $P^2+Q^2=\sqrt[3]{AB}$ (*702*) $=\frac{1}{3}a$ (*699*) and $P^3-3PQ^2=K$ (*702*) $=\frac{1}{2}b$ (*699*), thus the equation transforms into $x^3-ax-b=0$, which is the equation of (*699*).

712. Corollary. The equation just found can also be expressed as $x^3-3ex-2K=0$. The roots of the equation **A** (*702*) are the halves of the roots of this equation (*280*).

713. Corollary. The equation (*699*) can also be formed out of the factors [*Wurzelgleichungen*] in (*710*). This account is found in Maria Gaetana Agnesi's, *Instituzioni analitiche ad uso della gioventu' Italiana T. I.* § 187.

714. Corollary. If, in (*710*), $m=n$, then $c=0$ in (*699*); consequently, $\frac{1}{2}b=\left(\frac{a}{3}\right)^{\frac{3}{2}}$. Moreover follows the equation $x^3=ax+\dfrac{2a}{3}\sqrt{\frac{1}{3}a}$ and its root, $2m=2\sqrt[3]{\frac{1}{2}b}=\dfrac{2a}{3}\sqrt{\frac{1}{3}a}$. A completely simple example of such a case can be made in rational numbers is the triple of a square is taken for a.

716. Corollary. If the root of a cubic equation becomes expressed as $\sqrt[3]{\frac{1}{2}b+c}+\sqrt[3]{\frac{1}{2}b-c}$, then c is either possible or impossible according to whether two of the roots are impossible or all three roots are possible (*701; 705*), and hence, the true cause of why it becomes impossible in the latter case reveals itself. Namely, the adopted expression of the root signifies all three roots, by virtue of the procedures (*709; 710*). However, it cannot possible signify three possible roots in the case that c is not impossible (*709*), for if c were possible, it would yield two impossible roots according to (*710*).

717. Scholium. The procedure described in (*699*) is called **Cardan's Formula**. Cardan reports it in his Algebra, which carries the title: *Artis magnae siuede regulis algebraicis liber unus*, which was published in Basel, 1570, together with his *opera nouo de proportionibus numerorum; motuum* etc. In

Chapter I, page 5 of this publication, he ascribes the discovery of the formula to SCIPIO FERREUS of Bononien. According to his expression it is called: *capitulum cubi, & rerum, numero aequalium*; which is, according to our manner of saying, $x^3 + gx = h$, viz. the old Algebraicists calling **rem** what we denote x. NICOLAUS TARTALEA had also arrived at this formula and imparted it to CARDAN, though without proof, which CARDAN had found and provided. Cardan's Formula expressed in letters, appears somewhat different than the form given (*699*), however, it can easily be brought into that form. DESCARTES had arrived at the formula which flows out of Cardan's in his *Geometry* (Book III, Page 92 of the Amsterdam Edition 1683; or the Franckfurt 1695) and FRANCISCUS VAN SCHOOTEN repeated it in the work *de cubicarum aquat. resolutione** and thereto enclosed in HUDDEN's *Analysis*: He believes that if, however, all roots are possible, they are not permitted [*habe nicht statt*], as is shown in the case of solving the division of an angle in three parts. The formula can be derived in various ways; in (*699*), I have followed Herr EULER's in his *coniectatione de formis radicum aeqationum Comm. Ac. Petrop.* T. VI. p. 216. §§ 3;4.†

718. Scholium. This apparent inconsistency [*Ungereimtheit*] (*708*) has given mathematical understanding [*Mathematikverstndigen*] many headaches. NEWTON says in *Arithmetica Universalis: de Aequationum reductione per divisores surdos*, page 209 of GRAVESAND's edition:

> If it [(the equation)] gives three possible roots, then they all behave neutrally [*verhielten sie sich alle gleichgltig*] to the coefficients of the equation, and are expressed by x without any disparity, hence none can be found due to the fact that expressing all three according to this law is not possible.‡

---

*[Orig. Note] In the addendum to his *Commentar. in Geom. Cartes.* in op cit. *Aufl.* 345. S. as well as the work *de organica sect. con. descriptione* Leyden [*Lugdunum Batavorum*] 1646.

†[Trans. Note] This document is available in full in its original Latin thanks to the Euler Archive at: http:://www.math.dartmouth.edu/ euler/docs/originals/E030.pdf.

‡[Trans. Note] Since I feel I can clarify the German rendition of this no further without damaging the succinctness with which Kästner presents it, I include here the revelant sections of an English translation of Newton's *Universal Arithmetik* done by Joseph Ralphson, published in 1761:

> **CLII.** Yet there are some Reductions of *cubick Equations* commonly known, which, if I should wholly pass over, the Reader might perhaps think us deficient. *Let there be proposed the cubick Equation* $x^3 + qx + r = 0$, *the second Term whereof is wanting. For that every cubick Equation may be reduced to this Form, is evident from what we have said above. Let* x *be supposed* $= a + b$. *Then will be* $a^3 + 3aab + b^3$ (*that is* x^3) $+qx + r = 0$. *Let* $3aab + 3abb$ (*that is,* $3abx$) $+qx$ *be* $= 0$, *and then will* $a^3 + b^3 + r$ *be* $= 0$. *By the former Equation* b *is* $= -\frac{q}{3a}$, *and cubically* $b^3 = -\frac{q^3}{27a^3}$. *Therefore, by the latter,* $a^3 - \frac{q^3}{27a^3} + r$ *is* $= 0$, *or* $a^6 + ra^3 = \frac{q^3}{27}$, *and by the Extraction of the adfected quadratick Root,* $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. *Extract the cubic Root and you will have* a. *And above,*

It is easily seen that this says nothing: for even therefrom the same manifold of the roots arises, such that various unknown magnitudes, contrary to the given ones through which the coefficients become determined, all have the same manner, whether these unknown magnitudes are all possible, or a couple of these are impossible, just as example (*242*) demonstrates. Meanwhile, this position of NEWTON's prompted FRIEDRICH WILHELM STÜBNER to find here an application of the Principle of Sufficient Reason, since there was no existing reason as to why any one of the three roots should present itself more readily than any of the others (in his 1733 disputation held at Leipzig *contra virium mensuram cartesianam pro leibnitiana* § 134). Herr CLAIRAUT (in MYLIUS's German translation[§] of his *Algebra*, Parts V. VI. and further § 302. S.) also treated this formula and noticed that both the impossible cubic roots could, perhaps, still yield a possible sum, since the impossible magnitudes could cancel themselves out. He proceeds to transform each of the impossible cubic roots into an infinite series according to NICOLE's instruction (*Mem. de l'Ac. des Sc.* 1738, p 99-100), and shows that the impossible part, which is to be arranged separately in each will cancel by addition. This uncommonly laborious procedure is entirely superfluous if the matter is treated according to (*702; 709*), for them alone, however, some doubts still remain.

719. Scholium. For it does not follow, that a magnitude is possible, just

you had $-\frac{q}{3a} = b$, *and* $a + b = x$. *Therefore* $a - \frac{q}{3a}$ *is the Root of the Equation proposed*).

...

CLIII. *And after this Way the Roots of all cubical Equations may be extracted wherein* q *is Affirmative; or also wherein* q *is Negative, and* $\frac{q^3}{27}$ *not greater than* $\frac{1}{4}rr$, *that is, when two of the Roots of the equation are impossible. But where* q *is Negative, and* $\frac{q^3}{27}$ *at the same Time greater than* $\frac{1}{4}rr$, $\sqrt{\frac{1}{4}rr - \frac{q^3}{27}}$ *becomes an impossible Quantity; and so the Root of the Equation* x *or* y *will, in this Case, be impossible, viz. in this Case there are three possible Roots, which all of them are alike with respect to the Terms of the Equation* q *and* r, *and are indifferently denoted by the Letters* x *or* y, *and consequently all of them ought to be extracted by the same Method, and expressed the same Way as any one is extracted or expressed; but it is impossible to express all three by the Law aforesaid. The Quantity* $a - \frac{q}{3a}$ *whereby* x *is denoted, cannot be manifold, and for that Reason the Supposition that* x, *in this Case wherein it is threefold, may be equal to the Binomial* $a - \frac{q}{3a}$, *or* $a + b$, *the Cubes of whose Terms* $a^3 + b^3$ *may together be* $= r$, *and the triple Rectangle* $3ab$ *be* $= q$, *is plainly impossible; and it is no Wonder that from an impossible Hypothesis, an impossible Conclusion should follow.*

[§]Christian Mylius (1772-1754), a student of Kästner's and cousin to Gotthold Lessing, was part of a translating project that was most likely organized by Kästner, of which the cited translation would be a product. Mylius was a key collaborator of Kästner in the publication of the *Hamburgisches Magazin*, before Kästner left Leipzig for Göttingen. Mylius died mysteriously while makeing a stop over in England, en route to America where he was intended to make contact with Benjamin Franklin, amongst others.

because it can be expressed by a series in which all the terms that might be found are possible. For an increasing series can be found for $\sqrt{1-u}$ (*625*). The terms of this series, in so far as they can be found, are all possible regardless of whether the magnitude of u may be smaller or larger than 1. However, in the latter case, the series expresses an imaginary number, and yet it expresses such through purely possible terms. It is true that the series does not, thereupon, approach the value of the root, which it should express, but rather diverges from it (*12*), however, this does not demonstrate that it expresses something impossible. For the series in (*13*), for example, expresses the possible value of $\frac{1}{1+x}$; where x can be smaller or larger than 1, and yet, in the latter cases it diverges from the true values. If this very series contains purely possible terms, and yet signifies some impossible, then its supplement [*Ergänzung*] (*11*) will be impossible. Thus, although the series, which according to the procedure of NICOLE is the sum of two series with impossible terms, contains purely possible terms, yet it still must be investigated how it appears with the supplements [*Erg"anzungen*], and, were the terms impossible, whether they would also cancel themselves out.

720[a].[¶]**Scholium.** Incidentally, LEIBNIZ had already in 1696 (see his letter to WALLIS *Wallis. Opera Vol. III, coll. letters, letter 27*) made an observation [*Erinnerung*] just like that of NICOLE, namely, that the impossible magnitudes in the sum, as he says, *virtualiter* cancel themselves out; thus, NICOLE mistakenly boasts to have been the first to see this. KÖNIG (*Mem. de l'Ac. de Prusse 1749*, p.180) regarding this appearance of an impossibility, which he held to be something real, wished to account for it entirely with Logic, just as Stübner explained it with Metaphysics, such that, thus there only lacked a philosopher who would resort to Ethics for help. KÖNIG shows, namely, that the common analysis, through which the formula is found, presupposes something which leads to an impossibility if all three roots are possible. He thus hopes that an improvement of the analysis would disclose a formula, whereby this inconsistency [*Ungereimtheit*] would not occur. However, HARRIOT, WALLIS (*Algebra* 45;46; Cap. Op. Vol. II), EULER, and LANDEN (*Mathem. Lucubr.* P. VI) have each arrived at just the same formula by a suitable analysis, and shed light upon (*716*) such that nothing else can be brought out.

721[a]. Corollary. In the case of (*710*), m and n (*699*) can be found through an approximation.

Example. $x^3 = 6x - 40$; Here $a = 6$; $b = -40$; thus, $c^2 = 392$, and $x = \sqrt[3]{-20+\sqrt{392}} + \sqrt[3]{-20-\sqrt{392}}$ where the cubic root is extracted by approximation after beginning with the square root, and thus a nearly true value of x could be found.

[¶] [Trans. Note] For the following enumerations the symbols '[a]' and '[b]' have been appended to compensate for the error in enumeration that occurs in the original text.

720[b]. Corollary. Were, however, the actual value of x rational, then it would not be found with this procedure.

For example, $392 = 98 \cdot 4 = 14 \cdot 7 \cdot 4 = 2 \cdot 7 \cdot 7 \cdot 4$, thus $\sqrt{392} = 14\sqrt{2}$. Now, however, $(-2 \pm \sqrt{2})^3 = -20 \pm 14\sqrt{2}$ as can be affirmed by trial, save that I must necessarily show here how it will be found. Thus, $x = -2 + \sqrt{2} - 2 - \sqrt{2} = -4$.

721[b]. Corollary. In (*708*), however, where the approximation (*719*) could not be applied, there must be a method, provided the formula should be practical, to extract the cubic root out of $K \pm \lambda$ (*702*), or to find P and Q (*702*). The method employed in (*702*) onward, will not serve to this end since it directs to find P out of an equation, which is no more simpler than the last, except that the root is to be found through Cardan's Formula (*712*).

722. Corollary. A way analogous to the procedure in (*702*) could be sought in which the cubic root is picked out of a two-part magnitude, with one part rational, the other irrational, yet still being possible. This cubic root must assume an expression, which consists of a rational part and an irrational part, cubing both sides, and setting the rational equal to the rational, and also the irrational equal to the irrational. However, then both parts of the adopted expression which are sought, are also still left to be found through a cubic equation, for whose solution Cardan's Formula must be employed for such calculation. Thus, Wolf, *El. Analysis* § 360, to find the cubic root of $20 + \sqrt{392}$, solves the equation $z^3 - 6z = 40$, for the value of z, which he finds is 6 by some trials, and thereupon he finds the sought cubic root $2 + \sqrt{2}$. With this cubic root, he hence wishes to know if it is used to solve the equation $x^3 - 6x = 40$, just as Cardan's Formula gives it. Could he not have directly found the value of x in this equation just as easily?

723. Scholium. The analysts have sought methods to extract roots of every degree out of a two-part magnitude, whose one part is rational, the other irrational, for here cubic roots are especially necessary. Newton, *Arithmetica Universalis: de reductione radicalium ad simpliciores*; cet. p. 50, has provided instructions respecting such, although without proof, therewith various mathematical minds [*Mathematkverständigen*] have employed themselves, including Colin MacLaurin, *Algebra* P. I. §§ 118 onward, and Clairaut *Alg.* IIII. Th. XXV. , among other works. These treatments demanded, however, so much extensiveness that, on account of this, various authors, even Agnesi in her rather complete introduction to algebra, have omitted them. Attempts are still frequently provided, as with Colin MacLaurin *Alg.* P. I. § 131., which performs what is here regarded to be accomplished by (*721*), satisfying itself to designate a path, which is not wholly based upon experiments. Before the methods needed to treat the cases (*708; 721[b]*) were known, they had been called *casum irreducibilem*, which, according to Leibniz's report (epistle to Oldenburg; in the *Collected Epistles of Wallis, Opera* Vol. II), Raphael Bomelli was the first who taught how to find the roots of this seemingly impossible magnitude. They will excuse me, if I therefore do not completely furnish them here. I would have omitted Cardan's Formula, which according to the credentials of much mathematical understanding, including that of AGNESI, op cit. § 180, is of very little use, were the numerous and peculiar efforts which the Algebraicists have exerted upon a difficulty which is non-existent, not remarkable. Perhaps, after we have convenient methods to find the rational and irrational roots, their theory, which they themselves would not employ for actual application, just as I have reported it here, also will not be disagreeable. LANDEN, op cit., still has taken on the effort to find such a formula for the cubic equation where the second term is not absent, whereupon I must judge the same as in (*513*). Similar formulas for the biquadratic equation have been sought, and Herr EULER has established a conjecture based for this formula, as well as for the root of the general equation appearing surmountable (see *717*). I have thoroughly examined the preceding considerations in a Programme[‖] issued here in 1757, *Cardani sequationum cubicarum radices omnes tenere cet.*[**]

[‖] [Trans. Note] *Programme* refers to the annual public lecture which Kästner had been commissioned by Münchhausen to host every Easter at Göttingen, beginning 1756.

[**] [Trans. Note] See *GGA* April 14, 1757.