# Excerpts from *Anfangsgründe* Part 3 Section 1 *Analysis of Finite Magnitudes*

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**78. Task.** To solve an impure quadratic equation.

**Solution.** Their general expression is **G**) $z^2 = pz + q$, where $p$ and $q$ could be placed either positive or negative. The square has no coefficients, since if it did also, then they could be removed by division (*55*). One derives from this equation **H**) $z^2 - pz = q$. By adding the square of half the coefficient with $z$, or $\frac{1}{4}p^2$, to both sides of **H**, then one would have $\left(z - \frac{1}{2}p\right)^2 = z^2pz + \frac{1}{4}p^2$ (*Arithmetic II. 5.*). Then one obtains

**I**) $z^2 - pz + \frac{1}{4}p^2 = \frac{1}{4}p^2 + q$ or $\left(z - \frac{1}{2}p\right)^2 = \frac{1}{4}p^2 + q$, and consequently,

**K**) $z - \frac{1}{2}p = \pm\sqrt{\frac{1}{4}p^2 + q}$, and,

**L**) $z = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + q}$

Then **G** and **L** give the general formulae, which can be applied to all particular cases, if $p$, $q$ are properly specified.

. . .

**226. Corollary.** If an equation is divided by one of its factors [*Wurzelgleichungen*], then the division goes to completion, and the next lower comes out, which, when multiplied by the factor [*Wurzelgleichung*] produces the equation.

$$
\begin{array}{r|l}
 & x^3 - 5x^2 - 113x + 693 \\
\hline
x - 2 & x^4 - 7x^3 - 103x^2 + 919x - 1386 \\
 & -x^4 + 2x^3 \\
\cline{2-2}
 & -5x^3 - 103x^2 \\
 & 5x^3 - 10x^2 \\
\cline{2-2}
 & -113x^2 + 919x \\
 & 113x^2 - 226x \\
\cline{2-2}
 & 693x - 1386 \\
 & -693x + 1386 \\
\cline{2-2}
 & 0
\end{array}
$$

**227. Corollary.** Similarly, an equation $H$, which is a higher product of two others [$F$, $G$], when divided by $F$, results in the other, $G$;

Let $F = x^3 - 5x^2 - 113x + 693$, $G = x^2 - x - 2$, $H = x^5 - 6x^4 - 110x^3 + 816x^2 - 467x - 1386$.
Then,

$$
\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x^3 - 5x^2 - 113x + 693 & x^5 - 6x^4 - 110x^3 + 816x^2 - 467x - 1386 \\
 & -x^5 + 5x^4 + 113x^3 - 693x^2 \\
\cline{2-2}
 & -x^4 + 3x^3 + 123x^2 - 467x \\
 & x^4 - 5x^3 - 113x^2 + 693x \\
\cline{2-2}
 & -2x^3 + 10x^2 + 226x - 1386 \\
 & 2x^3 - 10x^2 - 226x + 1386 \\
\cline{2-2}
 & 0
\end{array}
$$

## Observations on the Irrational Roots of Equations.

**228. Theorem. The powers of an irrational number, whose square is rational, as $\sqrt{a}$, are rational for even powers; irrational for odd powers.**

**Proof.** If $n$ denotes a whole number, then $(\sqrt{a})^{2n} = (a^{\frac{1}{2}})^{2n}$ (*Arithmetic II. 9, 8.*) $= a^n$ is a power of the rational number $a$, and, consequently, rational: However, $(\sqrt{a})^{2n+1} = (\sqrt{a})^{2n}\sqrt{a}$ (*Arithmetic II. 7.*) $= a^n \cdot \sqrt{a}$, [which is] irrational.

**229. Corollary.** If $c$ denotes a rational number, then in any power of $c + \sqrt{a}$, the first, third, and the rest of the uneven terms are rational, and the second, fourth, and the rest of the even terms are irrational. This is clear from

(*141*): setting $c$ and $\sqrt{a}$ in the place of $a$ and $b$.

**230. Theorem. The powers of an impossible number, whose square is possible, as $\sqrt{-e}$ (*36*), are possible for all even powers; impossible for all odd powers.**

**Proof.** Its square is $-e$; and every even power, of exponent $2n$, is the $n$-th power of its square. For example, $n = 3$ is the cube of its square. Thus, all even powers are possible. Every odd power is a product of the immediately preceding even power, which was possible, with the impossible magnitude itself; for example, $(\sqrt{-e})^5 = (\sqrt{-e})^4 \cdot \sqrt{-e}$ and is thus, impossible, since a product of possibles into the impossible, which, according to the concept of multiplication (*Arithmetic I. 8.*), contains the impossible factor contains just as the possible contains the unit, therefore, it contains itself, and is consequently impossible.

Moreover, $(\sqrt{-e})^{2n} = ((\sqrt{-e})^2)^n$ (*Arithmetic II. 7.*) $= (-e)^n$, is positive or negative according whether $n$ is even or odd. However, the square of $(-e)^n$ is always positive and $= +e^{2n}$ despite $(-e)^n$ being negative as well (*31*). Further, $(-e)^{2n+1} = [(-e)^n \cdot \sqrt{-e}]^2$ (*Arithmetic II. 7.*) $= \sqrt{[((-e)^2)^2 \cdot -e]}$ (*26*) $= \sqrt{(+e^{2^n} - e)} = \sqrt{-e^{2n+1}}$.

**231. Corollary.** In [the expansion of] $(c + \sqrt{-e})^n$, all odd terms are possible, and all even terms impossible. Moreover, the former contain purely even powers of the impossible magnitude, the latter purely odd powers, just as in (*229*).

**232. Corollary.** The theorems (*229; 231*) also hold true for $c + b\sqrt{a}$; $c + b\sqrt{-e}$, since the rational factor, $b$, changes nothing in the proofs.

**233. Lemma. If $\mathbf{c + b\sqrt{a}}$ is a root of an equation, whose coefficients are all rational, then the equation also has $\mathbf{c - b\sqrt{a}}$ as a root.**

**Proof.** Let the equation be $x^3 + px^2 + qx + r$ and, for brevity, let $\sqrt{a} = \pi$. Substituting $c + b\pi$ for $x$, then

$$\textbf{F)}\quad \begin{array}{lllll} (c+b\pi)^3 & = c^3 & +3c^2b\pi & +3cb^2\pi^2 & +b^3\pi^3 \\ p\cdot(c+b\pi)^2 & = pc^2 & +2cpb\pi & +pb^2\pi^2 & \\ q\cdot(c+b\pi) & = qc & +qb\pi & & \\ r & = r & & & \end{array}$$

Here, on account of the rational coefficients, the first and third columns, or the odd, are rational, whereas the second and fourth, or the even, are irrational (*232*). Now, could the sum of the rationals and irrationals not = 0, or, which is just the same as saying, if the irrationals where brought to the other side of the equality sign with opposite signs, then the rationals cannot be equal, since the latter can be measured with the unit, whereas the former cannot (*Arithmetic*

*IIII. 26.*). Thus, for the equation **F**; or, what can equally be substituted for it:

$$\textbf{G)}\quad \begin{matrix} c^3 \\ +pc^2 \\ +qc \end{matrix} \quad \left.\begin{matrix} +3c \\ +p \end{matrix}\right\} \cdot b^2\pi^2 \quad = \quad -\left\{\begin{matrix} +3c^2 \\ +2cp \\ +q \end{matrix}\right\} \cdot b\pi \quad -b^3\pi^3 \;;$$

what is placed on either side of the equality sign in equation **G**, must itself be $= 0$; therewith transforming into $0 = 0$. Thus, the rationals themselves $= 0$ and the irrationals likewise; or

$$0 = \quad -\left\{\begin{matrix} +3c^2 \\ +2cp \\ +q \end{matrix}\right\} \cdot b\pi \quad -b^3\pi^3 \;;$$

or, the sum of the odd columns in equation **F** is $= 0$; and also for the sum of the even columns.

If now $c - b\pi$ is substituted for $x$, then an equation **H** results, which therein differs from **F** only in that its even columns are the opposite magnitudes of the even columns in **F**, the odd columns, however, being entirely identical in both:

$$\textbf{H)}\quad \begin{matrix} (c-b\pi)^3 & = c^3 & -3c^2b\pi & +3cb^2\pi^2 & -b^3\pi^3 \\ p\cdot(c-b\pi)^2 & = pc^2 & -2cpb\pi & +pb^2\pi^2 & \\ q\cdot(c-b\pi) & = qc & -qb\pi & & \\ r & = r & & & \end{matrix}$$

Moreover, the even columns of equation **F** are the products of the given magnitudes into odd powers of $+b\pi$, and the even columns of equation **H** are the products of identical magnitudes into identical odd powers of $-b\pi$; that is, into magnitudes, which are opposite of the even powers of $+b\pi$ (*31*). However, the even columns of both equations are products of identical given magnitudes into identical even powers of $+b\pi$ or of $-b\pi$; that is, into magnitudes, which have identical signs (*31*). Thus, [as] the sum of the even columns of equation **F** is $= 0$; so must the even columns of equation **H** also have a sum $= 0$; and the sum of the odd columns in each equation is also $= 0$; that is, if equation **F** is true, then **H** must also be true.

It is clear, however, that the cubic equation, used as an example [for its] mere distinctness, with its determined dimension, has no bearing upon the proof [in question]. Moreover, the proof results purely from the character of the even and uneven columns, which in every equation, whatever degree it is, remain identical.

**233. Example.** Substituting the root $3 + \sqrt{11}$ for $x$ in $x^3 - 5x^2 - 8x - 2$, then results

$$\textbf{F)}\quad \begin{matrix} +27 & +27\sqrt{11} & +99 & +11\sqrt{11} & = 0 \\ -45 & -30\sqrt{11} & -55 & & \\ -45 & -8\sqrt{11} & & & \\ -2 & & & & \end{matrix}$$

where $27-45-24-2+99-55=0$ and $27-30-8+11=0$; thus, the equation still remains if $-\sqrt{11}$ replaces $+\sqrt{11}$ in it.

**234. Corollary.** An equation whose root is $c+b\pi$, can be divided by the quadratic $(x-c-b\pi)\cdot(x-c+b\pi)$ or $x^2-cx+c^2-b^2\pi^2$ (*226; 227*), in which all the coefficients are rational.

**235. Lemma. An equation whose root is $\mathbf{c}+\mathbf{b}\sqrt{-\mathbf{e}}$, also has the root $\mathbf{c}-\mathbf{b}\sqrt{-\mathbf{e}}$. The proof is derived from (*231*), just as (*232*) derives from (*229*), and becomes entirely the same as (*232*) merely by placing possible and impossible [magnitudes] in the stead of rational and irrational [magnitudes].**

**236. Corollary.** It can be divided by the equation $(x-c-b\sqrt{-e})\cdot(x-c+b\sqrt{-e})$ or $x^2-2cx+c^2+b^2e$, in which all the coefficients are possible (*226; 227*).

...

**282. Task.** To remove the second term out of an equation.

**Solution.** An equation in which the second term is absent can be made out of the equation $P$ (*221*). Were $T$ (*281*) constructed such that the second term disappears, then $me+p=0$; $e=-\frac{1}{m}p$; thus $y-\frac{1}{m}p$ must be substituted for $x$.