Excerpts from *Anfangsgründe*
Part 3 Section 1
Analysis of Finite Magnitudes

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1760

153. Task. To find the proportion of the side AB of an equilateral triangle ABC (*Fig. 1*) to the radius KA of the circle in which it is described.

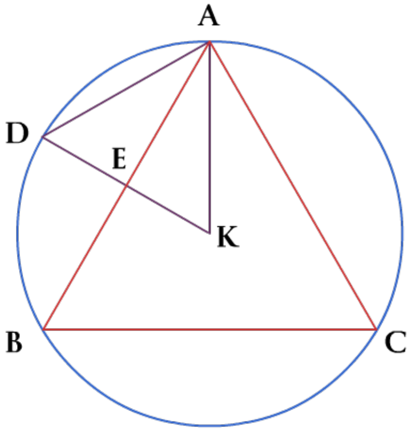

Figure 1

Solution. The arc ADB is 120^o (*Geometry. Theorem 23. Corollary 2.*). Dropping KED perpendicular upon AB, then $DKA = 60^o$, and $AE = \frac{1}{2}AB$ (*Geometry. Theorem 22. Corollary 2.*). Also $AD = AK = DK$ (*Geometry. Theorem 23. Corollary 5.*), and consequently, $DE = EK$ (*Geometry. Theorem 9. Corollary 5.*) $= \frac{1}{2}AD$. Therefore, $AE^2 = AD^2 - DE^2$ (*Geometry. Theorem 15.*). Setting the radius $= r$, the sought side $= x$, then $AE = \frac{1}{2}x$ and $AE^2 = \frac{1}{4}x^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$ or $x^2 = 3r^2$. Therefore, $x = \sqrt{3r^2} = r\sqrt{3} = r \cdot 1.7320508$ (*Arithmetic IV. 27*).

154. Corollary. $AE = 0.8660254$. Moreover, $\sin 90^o : \sin 60^o = r : AE$ (*Trigonometry. Definition 2.*) or $AE = \dfrac{r \cdot \sin 60^o}{\sin 90^o}$. Thus, supposing r has as many parts as the whole sine of the table, then AE becomes the sine of 60^o as stated in the table. For example, 8660254 here, if $r = 10000000$.

155. Scholium. The side of the polygon of $2n$ sides, z, is found from the side of the polygon of n sides, f, (*Geometry. Theorem 43.*), by $z^2 = 2r^2 - 2r\sqrt{r^2 - \frac{1}{4}f^2}$. Consequently, $2r\sqrt{r^2 - \frac{1}{4}f^2} = 2r^2 - z^2$ and by squaring both sides, $4r^4 - r^2f^2 = 4r^4 - 4r^2z^2 + z^4$ or $4r^2z^2 - z^4 = r^2f^2$ or $z\frac{\sqrt{4r^2-z^2}}{r} = f$, which then serves to find the side of the polygon of $2n$ sides from the side of the polygon of n sides.

Were one to set $z = r$, then one would hereupon immediately find f, [to be] the side of the triangle.

156. Task. To find the proportion of the side of the decagon $AB = x$ (*Fig. 2*) to the radius $AK = r$.

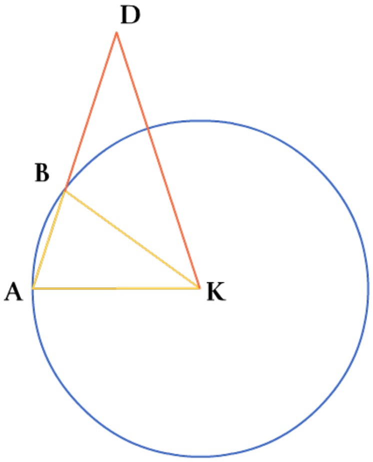

Figure 2

Solution. Were one to extend the side [AB], then, if KD is drawn at will, $BKD + BDK = ABK$ (*Geometry. Theorem 13. Corollary 1.*). Now, $AKB = \frac{4\mathbf{R}}{10} = \frac{2\mathbf{R}}{5}$, therefore, $ABK = \frac{4\mathbf{R}}{5}$ (*Geometry. Theorem 13. Corollary 5.*) $= 2AKB$.* Therefore, taking $BD = BK$, then $BKD = BDK = AKB$ and, due to the mutual angle at A, $\Delta ABK \propto \Delta AKD$ (*Geometry. Theorem 27.*). (I label the vertices of the triangles in the same order in which each equal angle stands in its triangle. For example, the angle K in triangle ABK is equal to the angle D in the triangle AKD.) Therefore, $AK : AB = AD : AK$ or, $r : x = r + x : x$ or $r^2 = x^2 + rx$ or $x^2 = -rx + r^2$. Instead of the aforementioned z, p, q, [found] in (*78*), here x, $-r$, $+r^2$, are substituted, giving $x = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + r^2}$. Here, the negative sign would yield something negative for the root, though doubtless the positive [root] is to be regarded as the sought side. Therefore, only the positive sign is employed, and this gives $x = \sqrt{\frac{5}{4}r^2} - \frac{1}{2}r$.

157. Prepatory Drawing. [*Verzeichnung*] (*constructio geometrica*) Drawing $CG = r$ perpendicular to $CF = \frac{1}{2}r$ [(*Fig. 3*)], and describing about F the arc GH with $FG = \sqrt{\frac{5}{4}r^2}$, (*Geometry. Theorem 15.*), thus $CH = FH - FC = x$.

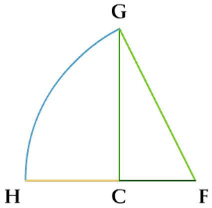

Figure 3

158. Corollary. While $x = \frac{r\sqrt{5}}{2} - \frac{1}{2}r = r\frac{\sqrt{5}-1}{2}$, then

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* [Trans. note] Kästner defines a right angle [*rechte Winkel*] in (*Geometry. Definition 11.*). Later, in the propositions, he introduces the symbol '$\mathbf{R}$' to denote a right angle (*Geometry. Theorem 6. Scholium 2.*).

$\sin 18^o = r\dfrac{\sqrt{5}-1}{4}$.

159. Corollary. If, respecting the side of the pentagon, following (*155*), $z = x$ (*156*) $= r\dfrac{\sqrt{5}-1}{2}$, then $4r^2 - z^2 = 4r^2 - \dfrac{5-2\sqrt{5}+1}{4}r^2 = \dfrac{r^2(5+\sqrt{5})}{2} = \dfrac{r^2(\sqrt{5}+1)\sqrt{5}}{2}$. Consequently, for the side of the pentagon $f = \dfrac{r(\sqrt{5}-1)}{2} \cdot \dfrac{r\sqrt{(\sqrt{5}+1)\sqrt{5}}}{2r} = \dfrac{r\sqrt{(\sqrt{5}+1)(\sqrt{5}-1)\sqrt{5}}}{4}$ (*28*) $= \dfrac{r\sqrt{(5-1)\sqrt{5}}}{4}$ (*Arithmetic II. 5.*) $\dfrac{r\sqrt{5}}{2}$ (*24*) $= x + \frac{r}{2} = FH$ (*157*).[†]

160. Task. To find the side of the octagon in a given circle.

Solution. The side of the square is $r\sqrt{2} = f$ (*155*), therefore, as in that place $z^2 = 2r^2 - 2r\sqrt{r^2 - \frac{1}{2}r^2} = 2r^2(1 - \sqrt{\frac{1}{2}})$.

161. Corollary. Moreover, by (*153, 155, 156, 159, 160*), the polygons of 3; 6; 12; 24; etc., of 4; 8; 16; 32; etc., and of 5; 10; 20; etc. sides can be constructed in the circle, and their sides be measured by the radius.

162. Corollary. Thence, the sines of half the center angles [*Mittelpunctswinkel (Geometry. Theorem 23. Corollary 3.)*] of these polygons is produced in the same manner as in (*154*).

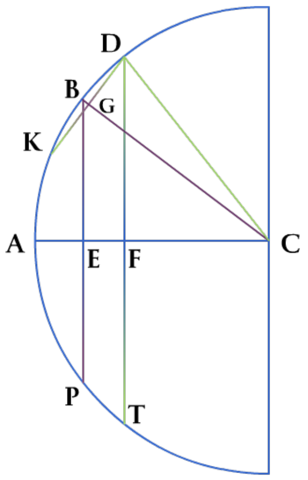

Figure 4

163. Task. If $BP = k$, $DT = c$, the parallel chords of the two arcs BAP,

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[†] [Trans. note] This conclusion is incorrect. First, the denominator of 2 is brought outside the root mistakenly. Second, though Kästner correctly develops the law for multiplication of roots [*Wurzelgrössen*] in the section he cites, (*28*), it is not applied correctly here. The calculation should be as follows: $f = \dfrac{r(\sqrt{5}-1)}{2} \cdot \dfrac{\sqrt{\dfrac{(\sqrt{5}+1)\sqrt{5}}{2}r^2}}{r} = r\sqrt{\dfrac{(\sqrt{5}+1)(\sqrt{5}-1)^2\sqrt{5}}{8}} = r\sqrt{\dfrac{(5-1)(\sqrt{5}-1)^2\sqrt{5}}{8}} = r\sqrt{\dfrac{5-\sqrt{5}}{2}}$. Thus, as it should correspond, the difficulty found in geometrically constructing the pentagon is more fittingly represented by this expression (one which Johannes Kepler classifies as being of the 8th Degree of Knowability).

DAT [*Fig. 4*], are given, find from there the chord of their difference, which shall be called h.

Solution. $BE = \frac{1}{2}k$, $DF = \frac{1}{2}c$ are the respective sines of half of the arcs BAP, DAT, and DG is the sine of their difference. The relationship [*Vergleichung*] between these 3 sines was found in (*Trigonometry. Theorem 19.*). So, $DAT - BAP = BD + PT = 2BD$, and since the difference of the arcs which belong to the chords is, once again, as great as the difference of the arcs to which the sines belong, it follows that DG is the sine of half the difference of the arc to which the chord belongs, and therefore $2DG = h$. Thus, replacing those [variables] in the cited (*Trigonometry*), $s = \frac{1}{2}c$, $m = \frac{1}{2}k$, $d = \frac{1}{2}h$, then $\frac{1}{2}h = \dfrac{\frac{1}{2}c\sqrt{r^2 - \frac{1}{4}k^2} - \frac{1}{2}k\sqrt{r^2 - \frac{1}{4}c^2}}{r}$, and thus, $h = \dfrac{c\sqrt{4r^2 - k^2} - k\sqrt{4r^2 - c^2}}{2r}$.

164. Corollary. Similarly, by (*Trigonometry. Theorem 19. II.*), is found the chord of the sum $c = \dfrac{h\sqrt{4r^2 - k^2} - k\sqrt{4r^2 - h^2}}{2r}$.

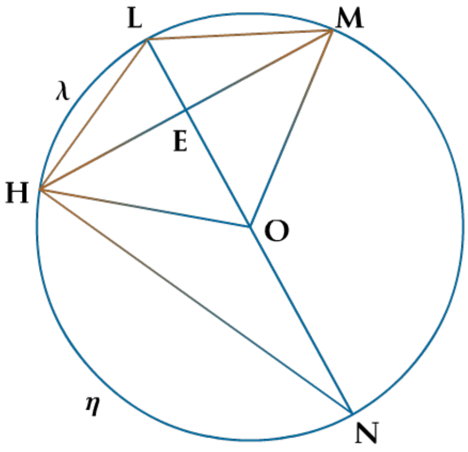

Figure 5

165. Corollary. If, in (*Fig. 5*), the arc $H\lambda L = \frac{1}{2}HLM$, then in (*163*) one sets $h = k = HL$, $c = HM$. That is, the chord HL can be regarded as the difference of the chord HLM and one which is as great as $H\lambda L$. If $r = 1$, this gives $2h = \dfrac{c\sqrt{4 - h^2} - h\sqrt{4 - c^2}}{2r}$ (*Trigonometry. Theorem 19. Scholium.*), and therefore, $h(2 + \sqrt{4 - c^2}) = c\sqrt{4 - h^2}$. If one squares the sides, then one finds finally, $h^2 = \dfrac{c^2}{2 + \sqrt{4 - c^2}}$, for h, the chord of the half arc, if c is given as the whole chord.

166. Corollary. Since the double arc is the sum of two single arcs, then with respect to (*164*) one sets $h = k$, $r = 1$. This yields $c = h\sqrt{4 - h^2}$ as the chord of the double arc, by way of the chord the single arc.

167. Corollary. If h in (*166*) is sought, we have $c^2 = 4h^2 - h^4$ or $h^4 = 4h^2 - c^2$, then, consequently, (*78*), $h^2 = 2 \pm \sqrt{4 - c^2}$. This gives two different values for the square of the chord h of the single arc in terms of the chord c of the double arc. That is, the chord HM (*Fig. 5*) belongs equally to the arcs HLM and HNM. Thus, halving the arc whose chord is HM, signifies halving each of the named arcs. If, however, $HLM = \omega$, the circumference $HLMNH = P$, then two arcs to which the chord HM belongs to are ω; $P - \omega$. Therefore, their halves $H\lambda L = \frac{1}{2}\omega$, $H\eta N = \frac{1}{2}P - \frac{1}{2}\omega$, which, together constitute $\frac{1}{2}P$. Consequently, both the values of h^2 appertain to both the squares of the chords HL, HN. If now $\omega < \frac{1}{2}P$, then $\frac{1}{2}\omega < \frac{1}{4}P$. However, $\frac{1}{2}P - \frac{1}{2}\omega > \frac{1}{4}P$, that is [respecting] the chords, $HL < HN$. In the case, then, that the smallest arc to which the chord c belongs is smaller than a quadrant, then the square of the chord of the half of the arc $= 2 - \sqrt{4 - c^2}$ and the square of the chord of half of its supplement to P [or, the remainder from P], $= 2 + \sqrt{4 - c^2}$. In the opposite instance, the case is turned around.

168. Scholium. This ambiguity, in virtue of which h can signify both HL as well as HN, though apparently eliminated in (*165*), in reality is still found to be there, since the radical quantity in the divisor can be regarded as positive as well as negative.

169. Scholium. The values from (*165; 167*) are also both equal, namely, if the divisor and dividend of (*165*) are multiplied with $2 - \sqrt{4 - c^2}$, then the magnitude remains unchanged, and its expression transforms itself into the expression in (*167*).

170. Scholium. The ambiguities are to be found just the same as one seeks the description of the sine or tangent of the half arc through (*Trigonometry. Theorem 19.*). (*167*) I have treated this generally for every division of the angle in my report: *Unde plures insint radices aequationibus sectiones angulerum definientibus*, Gött. 1756.

171. Scholium. Through the repetition of the task (*167*), one calculates the chords [*Sehnen*] for $\frac{1}{4}\omega$; $\frac{1}{8}\omega$; etc., or for all divisions, which might be found by halving, that the Algebra has here so far accorded with common Geometry (*Geometry. Theorem 5. Corollary.*).

172. Corollary. If one had the sides of two regular polygons of m and n angles, that is, the chords of [the arcs] $\frac{1}{m}P$ and $\frac{1}{n}P$, then the chord of $\dfrac{m-n}{mn}P$ is found by (*163*). Were $m - n$ now a power of 2, then the chord of $\dfrac{1}{mn}P$ is found by (*171*), that is, the side of the polygon which has mn angles.

Let $m = 5$, $n = 3$. Then the chord of [the arc] $\frac{2}{15}P$ is found from the sides of the pentagon and the triangle (*159; 153*), by whose halving the pentakaedecagon is found. See *Clavius Scholia ad Euclid IV. 16.*[‡] If the chord of

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[‡] The reference is to Christoph Clavius's *Commentaries on Euclid* (Rome 1589). The Jesuit, Clavius, was a contemporary astronomer of Johannes Kepler, who oversaw the Gregorian

the sum of $\frac{1}{3}P + \frac{1}{5}P$, that is, of $\frac{8}{15}P$, is sought, as in (*164*), then the pentakaedecagon would also be arrived at by three halvings.

173. Scholium. Polygons, outside of those which were found in (*161; 171*), can not be found by a solution to a quadratic equation and, moreover, are not registered with common Geometry, as will become more extensively shown below.

174. Task. Given the side of a regular polygon, find the radius of the circle which would describe it.

Solution. One need only examine the equation which gives the side of the polygon by the radius, taking the latter as the sought magnitude and the former as the given.

E. g., in (*156*), $\dfrac{2x}{\sqrt{5}-1} = r$.

175. Task. Given radius $= 1$, with sine $= a$ and cosine $= b$, for a single angle, find the sine and cosine of the multiple angle.

Solution. The multiple angle results by the single angle added to itself repeatedly. Therefore, with respect to [the formulas in] (*Trigonometry. Theorem 19.*), $d = a$, $\delta = b$. Further, for the double angle, $m = a$, $\mu = b$. Thus, for the double angle, the sine $= 2ab$ and the cosine $= b^2 - a^2$ (*Trigonometry. Theorem 19. Corollary 2.*). Now, setting in the place of m and μ, the sine and cosine of the double angle, then one has for the triple angle, sine $= a(b^2 - a^2) + b \cdot 2ab = 3ab^2 - a^3$, and cosine $= (b^2 - a^2) \cdot b - 2ab \cdot a = b^3 - 3a^2b$. In turn, using this in the place of m and μ, gives, for the quadruple angle, sine $= (b^3 - 3a^2b) \cdot a + (3ab^2 - a^3) \cdot b = 4ab^3 - 4a^3b$, and cosine $= (b^3 - 3a^2b) \cdot b - (3ab^2 - a^3) \cdot a = b^4 - 6a^2b^2 + a^4$. This, in place, handed down to m and μ, gives the quintuple angle sine $= (4ab^3 - 4a^3b) \cdot b + (b^4 - 6a^2b^2 + a^4) \cdot a = 5ab^4 - 10a^3b^2 + a^5$, and cosine $(b^4 - 6a^2b^2 + a^4) \cdot b - (4ab^3 - 4a^3b) \cdot a = b^5 - 10a^2b^3 + 5a^4b$; and so forth.

176. Corollary. The explicitly calculated sines and cosines observe the following law: If one raises $b + a$ to the power whose exponent signifies the multiplicity of the angle, and without concern for the sign of the terms of this power, but rather only regarding the magnitude yielded, then the sine is comprised of the terms of the second, fourth, sixth, etc. power, and the cosine is comprised of the terms of the first, third, fifth, etc. power; the former, the even terms, the latter the odd. The signs of the terms, however, alternate in each of the series for the sine and cosine, such that the first term is positive.

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calendar reform, involving the question of determining the date of Easter. The referenced passage here can be found in *Opera Mathematica of Christoph Clavius* pg. 159-165. The entire *Opera* is available online at http://mathematics.library.nd.edu/clavius.

An example is provided if the series for the fifth power of $b + a$, which was found in (*116*), is compared with the sine and cosine of the quintuple angle [as given here].

177. Corollary. $(b + a)^n$ can be constructed according to the Binomial Theorem (*141*), namely, by swapping what is there [denoted] a and b, and by replacing the m [found] there, wit the n [found] here. Moreover, n can be replaced with $4r$, such that P, therefore, would signify the coefficients whose exponent [*Zahl*] is $4r$, thus permitting a double division by 2 [that is, division by 4], (*numerus partier par*). Further, with respect to the coefficients immediately following, Q, R, S, T, the number for R would be $4r + 2$, which permits only a single division by 2; and respecting the coefficients at the beginning, A, B, C, D, etc., [we would have] $A = n$, $B = \dfrac{n \cdot (n-1)}{1 \cdot 2}$, $C = \dfrac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3}$, and so on. This then gives the observed law (*176*), when it is generally expressed for the n-tuple angle.

$$\begin{aligned} \text{Cosine} =& b^n - Bb^{n-2}a^2 + Db^{n-4}a^4 \ldots \\ &+ Pb^{n-4r}a^{4r} - Rb^{n-2r}a^{2r} + a = \mu \end{aligned}$$

$$\begin{aligned} \text{Sine} =& Ab^{n-1}a - Cb^{n-3}a^3 + Eb^{n-5}a^5 \ldots \\ &+ Qb^{n-4r-1}a^{4r+1} - Sb^{n-2r-3}a^{2r+3} = m \end{aligned}$$

Thence, it follow for the $(n + 1)$-tuple angle that $b\mu - am =$

$$\begin{aligned} \text{Cosine} =& b^{n+1} - (A + B)b^{n-1}a^2 + (C + D)b^{n-3}a^4 \ldots \\ &- (Q + R)b^{n-4r-1}a^{4r+2} + (S + T)b^{n-4r-3}a^{4r+4} \ldots \end{aligned}$$

and similarly $a\mu + bm =$

$$\begin{aligned} \text{Sine } =& (A + 1)b^n a - (B + C)b^{n-2}a^3 \ldots \\ &+ (P + Q)b^{n-4r}a^{4r+1} - (R + S)b^{n-2r-2}a^{2r+3} \ldots \end{aligned}$$

However, with respect to the $(n + 1)$ power, $A + B$, $C + D$, $Q + R$, and $S + T$, are the second, fourth, $(4r + 2)$-th, and $(4r + 4)$-th coefficients, in the same way that B, D, R, T, are the second, fourth, $(4r + 2)$-th, and $(4r + 4)$-th coefficients of the n power, by (*136*). Therefore, when the law is assumed for the cosine of the n-tuple angle, then it follows for the cosine of the $(n + 1)$-tuple angle. In the same way, the sine of the latter follows from the sine of the former, since $A + 1$, $B + C$, $P + Q$, $R + S$, and A, C, Q, S, are the first, third, $(4r + 1)$-th, and $(4r + 3)$-th coefficients of the $(n + 1)$ and n powers, respectively. Thus, if the assumed law is true for $n = 2, 3, 4, 5$, then it is true for all multiple angles; concluding in the manner which was used in (*139*).

178. Scholium. The tangents of the multiple angles can also be found in this manner, just as in (*Trigonometry. Theorem 19. Corollary 3.*). Their general law can also be determined, which can be accomplished even more readily by use of the Integral Calculus.