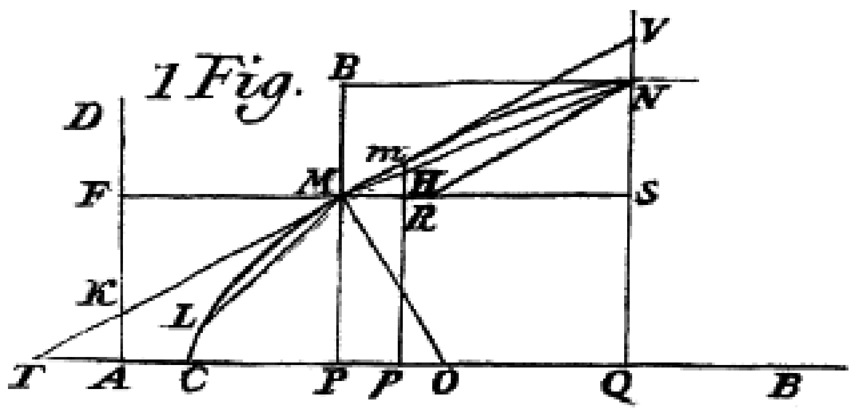
Excerpts from *Anfangsgründe* Part 3 Section 2 *Analysis of the Infinite*

Abraham Gotthelf Kästner.

1761



76. Definition. If the extended tangent intersects the abscissa at T, then PT is called the **subtangent**.

77. Task. To find the proportion of the ordinate to the subtangent, $MP : PT$.

Solution. $MP : PT = VS : SM$ (71) is also just as large or as small as the distance between VS and MP, if, however, one lets the first line infinitely approach the second, then the proportion $VS : SM$ will infinitely approach the proportion $NS : SM$ (75). Consequently, $MP : PT$ equals the proportion $NS : SM$ under the constraint that N is infinitely near M. However, since $NS : SM$, is the proportion of the differentials of the ordinate and the abscissa (14), thus calling $PM = y$; $AP = x$; then $PT = \frac{y.dx}{dy}$, which is found in finite magnitudes if one seeks according to the formula of the differential calculus of the quotients $\frac{dx}{dy}$. Thereupon it is also found that $AT = TP - AP$.

78. Corollary. If Mm denotes the element of the curved line (68), then $mR = dy$; $MR = dx$, follows out of the similarity of the triangles MmR and TMP. The triangle MmR, of which Barrow (s. 90) had already observed, Leibniz called *triangulum characteristicum* since through it curved lines can be differentiated.[1]

79. Corollary. If N approaches M infinitely, then NS becomes infinitely small, thus will VN be infinitely small in comparison with infinitely small magnitudes (74), or an infinitely small magnitude of the second order, if NS is one of the first order; or, if $NS = \frac{1}{\infty}$, then $VN = \frac{1}{\infty^2}$ (11). The argument is: by reducing VN and NS together, then VN will eventually become smaller than any assignable magnitude, since the magnitude of NS can be further indicated.

80. Corollary. In the figures out of which the calculation of 77 was based, it is assumed that the ordinates and the abscissae increase together from M to N, as well as decrease together from N to M, or that their differentials are both positive or negative (23); consequently, their quotient is positive and thus is the subtangent positive also, if the ordinate is assumed positive. At the same time, however, it is evident simply from the figures that the subtangent falls from P to the origin of the abscissae, or, T and A overlie a side of the ordinate.

81. Corollary. Yet, take the ordinate ab, in which the abscissa increases just as if B were the origin of the abscissae, then the quotient of their differentials would be negative, and thus the subtangent would also be negative with a positive ordinate. However, T and B lie on two different sides of the ordinate.

82. Corollary. Were a negative ordinate given, the subtangent in 80 would be negative, and positive in 81.

83. Corollary. The subtangent and the origin of the abscissae will lie on the same or differing sides of the ordinate depending on whether the subtangent and ordinate have the same or different signs.

Example to 77.

84. Task. To find the subtangent of the parabola (Alg. 349).

Solution. The equation for the parabola, $y^2 = bx$, will yield the differential equation $y = bdx$ (18;24) or $\frac{dx}{dy} = \frac{2y}{b}$; consequently $PT = \frac{2y^2}{b} = \frac{2bx}{b} = 2x$ or, the subtangent equals the double of the abscissa of the parabola if the abscissa is reckoned from the vertex.

Specification [*Verzeichnung*]. To draw a tangent at a given point M of the parabola, draw the ordinate MP and take AT equal to the abscissa AP, but backwards from the vertex, then TM will be the tangent.

85. Task. To find the subtangent for the ellipse (362) and the hyperbola (389).

[1] Translator's Note: The German is, "… *weil sich dadurch die krummen Linien unterscheiden lassen.*" Here '*unterscheiden*' holds for both meanings of differentiate, i.e. the generic (syn. distinguish between) and the specific, technical (e.g., to *differentiate* an algebraic function). Whether this ambiguity is merely modern, or intended by Kästner, is an open question, though the ambiguity in the least gives rise to a richer conception of this aspect of the calculus.

Solution. The equation $y^2 = bx \mp \frac{bx^2}{a}$, where the upper sign is true for the ellipse, the lower for the hyperbola, gives $2ydy = \left(b \mp \frac{2bx}{a}\right) dx$ or $\frac{dx}{dy} = \frac{2ay}{b.(a \mp 2x)}$; consequently, $\frac{y.dx}{dy}$ or $PT = \frac{2x(a \mp x)}{a \mp 2x}$. Thus $PT - x$ or $AT = \frac{ax}{a \mp 2x}$.

Specification [*Verzeichnung*]. To draw a tangent at a given point M of the ellipse or the hyperbola, draw its ordinate and seek the fourth proportional line to $a \mp 2x$; x; a; which erected from the vertex, yields the point T.

For the ellipse, the numerator is constantly positive, (Alg. 375) whereas the denominator is positive or negative, depending upon whether the abscissa as reckoned from the vertex is smaller or larger than the semi-axis. Thus, in the former case, the subtangent and origin of the abscissae lie on a single side of the ordinate, in the latter different, identical to how the figure of the ellipse is drawn. For the hyperbola, there are even more case to discern, since x increases without end, and can also become negative; each will, however, be determined from the figure and can be compared with 83.

86. Corollary. For $x = \frac{1}{2}a$ in the ellipse, $\frac{dy}{dx}$ or $b \cdot \frac{a - 2x}{ay} = \frac{b.0}{ay} = 0$. Thus, here dy vanishes in comparison with dx, that is, around the midpoint the ordinate increases infinitely less than the abscissa. Accordingly, the angle mMR, from 78, will be infinitely small, or the tangent mMT will parallel the line of the abscissae. Therewith is agreed upon that in this case $PT = a\frac{\frac{1}{2}a}{0}$ becomes infinite. Similarly, $\frac{dx}{dy} = \frac{ay}{b.0}$ becomes infinite, which says nothing else than that the reciprocal $\frac{dy}{dx}$ vanishes.

87. Corollary. Generally, if for a certain abscissa $\frac{dy}{dx} = 0$, then the tangent is parallel to the line of the abscissae, and, consequently, is perpendicular to the line which the line of the abscissae is perpendicular to, just as AD in figure 1. From here it is seen what it means for $\frac{dy}{dx} = \infty$. Thereupon $\frac{dx}{dy} = 0$ or the tangent is parallel to the line AD, which runs parallel with the ordinate y, and thus is perpendicular to the line of the abscissae x, if APM is a right angle.

88. Corollary. The formula in 85, can be transformed into that in 84 according to 10, if one sets $a = \infty$; on account of **Alg. 384**.

89. Corollary. These instructions also apply if the line of the abscissae is not equal to the axis, but rather is merely a diameter **Alg. 453**.

90. Scholium. The formula in 77 results in its entirety out of one which Barrow had given at the end of *Lect. Geometr. X.* as and appendix to *art. XIIII.*, excepting that he serves it in other expressions. Namely, one can call $AF = y$, the abscissa and $FM = x$, the ordinate. Newton had certainly read the paper of his teacher, Barrow, and probably Leibniz as well. Further, the first applications of the fluxions and the differential calculus were the drawing of tangents. This indication [*Errinnerung*] corresponds to 44. *Jac. Bernoulli* had already made it in his first *Specimine calculi differentialis (Acta Eruditorum, Leipsic, Jan. 1691)*; and, probably not to dissatisfy Leibniz, somewhat moderated [*gemildert*] at the end of the second *Speciminis (op cit, Jun. 1961)*. See *Opera Jacques Bernoulli* p. 431 and 453.

91. Definition. If MO stands perpendicular from the tangent, intersecting the line of the abscissae at O, then it is called a **normal** and PO the **subnormal**.

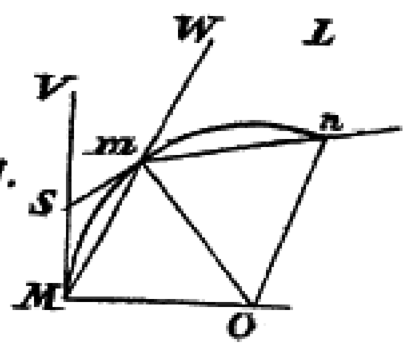
92. Corollary. If the ordinates make a right angle with the abscissae, then the subnormal $PO = \dfrac{PM^2}{PT} = \dfrac{y^2.dy}{ydx}$ (77) $= \dfrac{ydy}{dx}$, just as the similarity of the triangles MmR; MOP illuminates also.

93. Corollary. Thus for the **parabola (84)**, $PO = y\dfrac{b}{2y} = \frac{1}{2}b$, or constant, and always equal to the half-parameter.

94. Corollary. For the **ellipse** and **hyperbola (85)**, $PO = (a \mp 2x2a$.

95. Scholium. If $x = \frac{1}{2}a$ in the ellipse, then the subnormal will $= 0$, by reason of (37). If $b = a$, such that the ellipse transforms into a circle, then the subnormal of the circle $= \frac{1}{2}a - x = OP$, Fig. 1, if LMN were the arc of a circle around the center O, since there all the normals go through the center.

5 Fig.



96. Corollary. Let Mm, mn (Fig. 5), be two equal consecutive arcs of a circle around a center O; let them both be finite at the outset. Let MS be the tangent for M. Denote the angle $Mom = mOn$ with u; let the chord Mm be extended into W, and mn the other chord. Then $OMS = \mathbf{R}$; $OMm = OmM = Omn = \mathbf{R} - \frac{1}{2}u$ (Geometry. Theorem 13. Corollary 5.). Consequently, $Wmn = WmO - Omn = mMO + u - mMO = u$. Since $OmS = \mathbf{R}$, the tangents cut each other with the angle $VSm = u$. Now if the chords Mm; mn; of this arc infinitely approach, then the angle u will become infinitely small, and the theorem [*Satz*] will be expressed thus: If the circle is divided into equal elements, then the extension of each of the elements with that following makes an angle which is greater than the angle which corresponds to each of the elements at the center. The angle of an element with the radius, however, at half of the just mentioned anlge, differs from a right angle. If the infinitely small variations, or diminishing angles, are sought, then $OMm = \mathbf{R} - \frac{1}{2}u$ must be assumed; otherwise one can $OMm = \mathbf{R}$, from which it differs infinitely little.

97. Task. To draw a tangent of the conchoid [*Muschellinie*], Alg. 479.

Solution. In the equation given there, $(b+x)\sqrt{a^2 - z^2} = yz$, one sets $a^2 - x^2 = z^2$, thus $-\frac{xdx}{z} = dz$; furthermore, $(b+x)z = yz$, thus