Göttingen Notices of Learned Matters

under the supervision
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In the regular meeting of the Royal Society of the Sciences [Göttingen] on December 4, Professor KÄSTNER examined [the question:] around a sphere, how many other spheres, each of equal magnitude, could be placed in such a way that all would stay around the middle sphere and touch the neighboring spheres standing around themselves. The two points where two such neighboring spheres touch the middle sphere, must be separated from each other by an arc which will be determined by a magnitude which each of the surrounding spheres will have and, consequently, will be the same magnitude between all the points. Three such points of contact [*Berührungspuncte*], therefore, generate an equilateral spherical triangle. They belong to three adjacent surrounding spheres, all which mutually touch each other, and should this be done by every three neighboring spheres, then the entire spherical surface must become covered as with a net of such equilateral, similar triangles. Each of these triangles must therefore be an aliquot part of the spherical surface, which they divide, and furthermore they must meet together equally to form spherical angle-points [*Winkelpuncte*], and the sums of the angles which lie around the angle-points must produce 360^o. Hence, this investigation so arranged is founded on the calculation of the surfaces of the spherical triangles.

What JACOB BERNOULLI had already long since [*vorlängst*] taught regarding this, constituted the beginning of Hrn. KÄSTNER's investigation: [presenting] a formula, arranged more conveniently according to the modern usage, out of which was derived at once Hrn. EULER's theorem of the comparison [*Vergleichung*] of the surface of a spherical triangle with the sum of its three angles. Hrn. KÄSTNER wanted to directly present this theorem, which being so contracted scarcely fits a page, rather than require the reader to search for the origin of this in other books. From this theorem, and spherical trigonometry, the following ensues:

If the side of an equilateral spherical triangle $= \eta$; the angle x; the proportion of the diameter to the circumference of the circle is called $1 : \pi$; then, $\sin \frac{1}{2}x = \dfrac{1}{2\cos\frac{1}{2}\eta}$, and the triangle's surface $= 3x - \pi$. Should the triangle be contained in the surface of the sphere n times, then $x = \frac{1}{3}\pi \cdot \left(1 + \frac{4}{n}\right)$. Should, however, m angles of this triangle be equal to 360^o, then $n = \dfrac{4m}{6-m}$, wherefrom it follows that there are only four kinds of nets: of 2; 4; 8; and 20 triangles. The first becomes made by the great circle which halves the sphere. The triangles of these four kinds of nets give 3; 4; 6; 12 distinct points of contact, that therefore follows from this variety of these four cases, 3; 4; 6; 12 spheres which could be placed around a middle sphere in the required manner. One can easily conjecture from here that the last three nets become specified in the spherical surface by the polygons of the three regular solids, which are bounded with plane triangles, and the conjecture lets itself be brought to proof immediately. Hrn. KÄSTNER, meanwhile, has shone it naturally and generally that the matter immediately leads out from the consideration of the spherical triangles.

The radii of the surrounding spheres are all greater than that of the middle sphere. For the case of the twelve spheres, the radius of the middle sphere is to the radii of the surrounding spheres just as the excess of the side of the regular pentagon over the side of the regular decagon is to the side of the regular decagon; or the radii of the surrounding spheres are 1.1085 of the radius of the middle sphere. For fewer spheres, the radii are greater.

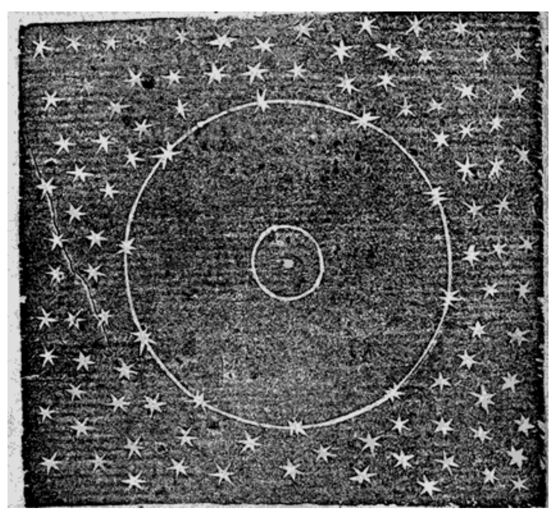

Figure 1: Original image from the section of Kepler's *Epitome* cited by Kästner.

One could also imagine a sphere which encloses within itself all the middle and the bordering spheres, and this taken as a [new] middle sphere for others to surround; thus extending the investigation further. Finally, Hrn. KÄSTNER pointed out how twelve spheres all equaling the middle, could become placed around, such that, so much as they might allow, every one would touch each other. KEPLER had given cause to Hrn. KÄSTNER for this investigation when he determines in his *Epitome of Copernican Astronomy, Book I, Part II*, that there must be a greater interval [*Zwischenraum*] between our Solar System [*Planetenwelt*] and the heaven of the fixed stars, while otherwise only some twelve fixed stars could be placed around our

Solar System [*Planetenwelt*], which as for the rest, would always recede further and further, and therefore must finally vanish before our eyes. He arrives at the number of the twelve fixed stars from the twelve vertices of the icosohedron, in which each is approximately as distant from the middle point, and it came to pass, thence, as the middle point is to the place of the fixed stars, if the fixed stars should be distributed in an similar manner in the space of heaven [*Himmelsraume*]. One sees that KEPLER aimed at a problem approximate to the present, although he did not investigate the geometrical part of it, which alone is capable of certain knowledge; however, astronomical conjectures do not belong here.

Otherwise, perhaps THOMAS WRIGHT could employ this investigation, who in his *New Theory of the Universe*, believes that the stars must be placed symmetrical around a definite point from which the eye of prudence, so to say, might examine them. They appear to us only to stand irregularly, because we do not view them from these proper vantage points. Incidentally it could also be objected against KEPLER, that indeed only 13 undoubtedly fixed stars of the first magnitude could be counted, of which, on account of the uncertain appraisal of these magnitudes, one could easily have been missed. The remaining will become noticed, not on account of their apparent size, but by the intensity of their light, and it is conceivable, if they also stand at very different distances from us, that we could notice no great difference in their apparent magnitude, where we are absolutely incapable of measuring their apparent magnitude.