Excerpts from *Anfangsgründe*
Part 3 Section 1
Analysis of Finite Magnitudes

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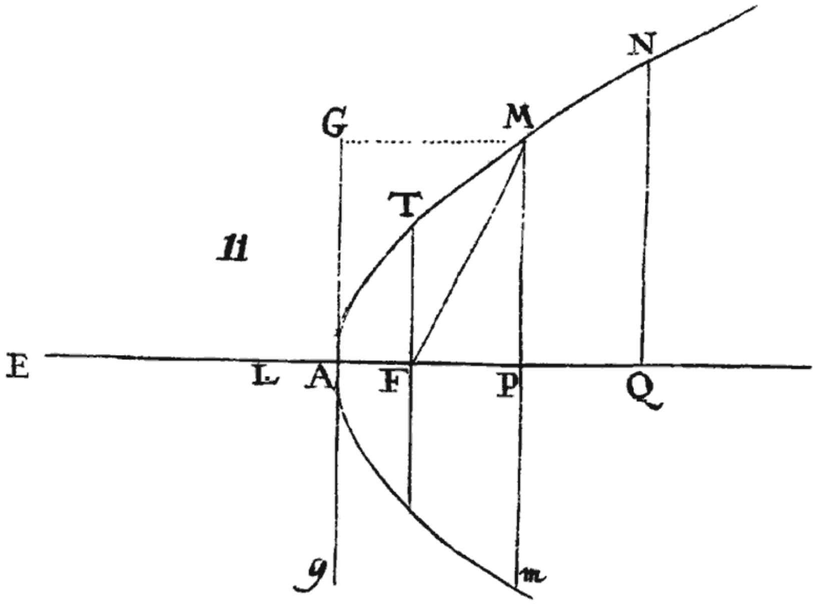
1760

The Parabola



349. Definition. The parabola is a curved line, where $ax = y^2$ using rectilinear coordinates (*331*). The constant line a is called the **parameter** and the ordinate is thus the mean proportional line between the parameter and the abscisse (*Arithmetic V. 32, 33.*).

350. Corollary. For every given abscisse AP (*Fig. 11*), and the constant parameter AE, the ordinate PM is found (*Geometry 31.*), such that it stands perpendicular to P on the axis AP (*331*) for which the position is known, but not the length. That is to say, one need only describe a half circle [not shown] on EP, which cuts off a perpendicular AG passing through A on EP. The perpendicular on P $[PM]$ is made equal to this perpendicular, which can be done by means of a parallel with the axis (*Geometry. Theorem 12. Corollary 3.*). Thus every point on the parabola can be determined by means of the common geometry.

351. Corollary. Since $y = \pm\sqrt{ax}$, two opposite but otherwise equal ordinates PM, Pm belong to every abscisse (*333*). The construction (*350*) likewise gives the same thing, because the half circle can be described both above and below the axis.

352. Corollary. The parabola is divided by the axis into two similar and equal halves, as the circle is divided by its diameter (*Geometry. Axiom 6.*).

353. Scholium. Hence, the ancients called the cord Mm of the parabola *ordinatas* and its half, PM, *semiordinatas.* I use these names in such a way that they can be used with all curved lines.

354. Corollary. Because $y = 0$ when $x = 0$, the axis of the parabola is cut at the beginning of the abscissa A. This place is called the **extremum of the parabola**.

355. Corollary. Because ax always remains positive for a positive x, and grows together with it, the ordinate, or $\sqrt{ax}$, grows endlessly with the abscissa and surpasses all limits, that is, the parabola extends on either side of the axis with infinite arms. These arms cross over all perpendiculars which are drawn to the axis, and over all parallels which can be drawn to the axis, so that the space through which the parabola extends on the side where its infinite arms lie can in no way be bounded. The arms proceed ever further along the axis, and also open ever wider from one another.

356. Corollary. For a negative x, ax would be negative, and y impossible (*34*). Thus, there are no ordinates for the negative abscissas, and the entire parabola lies on one side of the perpendicular AG.

357. Corollary. If QN is another ordinate, $a \cdot AQ = QN^2$, thus $QN^2 : PM^2 = AQ : AP$, and $QN : PM = \sqrt{AQ} : \sqrt{AP}$; or, in other words, the abscissas are as the square of the ordinates, and the ordinates as the square roots of the abscissas; (*abscissae sunt in ratione duplicata ordinatarum, et ordinatae in subduplicata abscissarum*).

358. Definition. The **focus**, F, is the place on the axis, where the ordinate FT is equal to the half parameter.

359. Corollary. Thus $a \cdot AF = FT^2 = \frac{1}{4}a^2$ or $AF = \frac{1}{4}a$.

360. Corollary. $FM = \sqrt{((x-\frac{1}{4}a)^2+y^2)} = \sqrt{(x^2+\frac{1}{2}ax+\frac{1}{16}a^2)} = x+\frac{1}{4}a$.

361. Corollary. To find points which are on a parabola that has AQ as axis, A as extremum, and a as parameter, one proceeds thus: on both sides of the extremum, extend $AF = AL = \frac{1}{4}a$. Thus, F is the focus (*359*). $AP = x$ is taken arbitrarily, and through P a perpendicular is erected on the axis. A circle whose radius is $= LP = \frac{1}{4}a+x$ is described around F. It cuts the perpendicular in M, m, two points on the parabola (*360*). In such a way both points are found for every point taken on the abscissa.

The Ellipse

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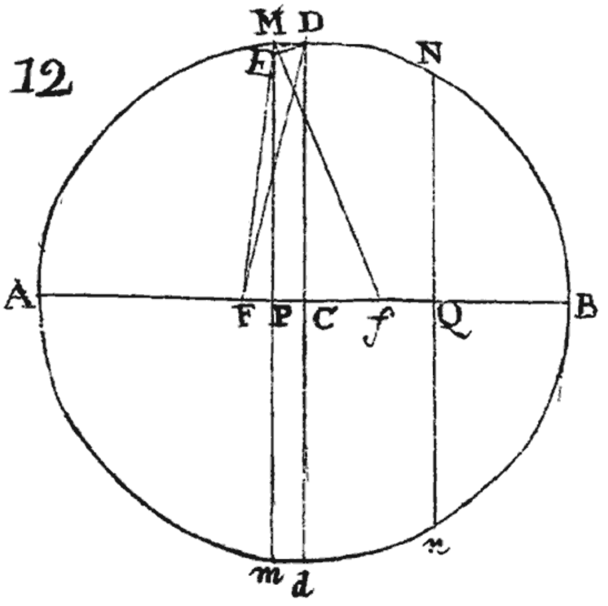


362. Definition. For the ellipse, in rectangular coordinates $y^2 = bx - bx^2 : a$; the constant magnitude b is called the **parameter**, and a the **major axis**.

363. Corollary. For $y = 0$, x can either equal 0 or a. Thus, if the abscissa is calculated from A, (*Fig. 12*), and $AB = a$, the points A, B, which are on the ellipse and on the axis simultaneously, are the extrema of the ellipse (just as in *354*) and their distance is the major axis.

364. Corollary. Because $y = \pm\sqrt{(bx - bx^2 : 2)}$, two opposite but otherwise equal ordinates belong to every abscissa, and the ellipse is divided into similar halves $AMDB$ and $AmdB$.

365. Corollary. For $x = \frac{1}{2}a = AC$ we have $y = \frac{\sqrt{ab}}{2} = CD$. The double of this line, Dd, is called the **minor axis** (*axis conjugatus*), and C the **center**.

366. Corollary. Let the semi-minor axis be $= \frac{1}{2}c$; therefore $c^2 = ab$ and thus $\dfrac{c^2}{a} = b$ or, the parameter is the third proportional line between the major and minor axes.

367. Corollary. Thus $y^2 = \dfrac{c^2x}{a} - \dfrac{c^2x^2}{a^2}$.

368. Corollary. If $CP = u$, then $x = \frac{1}{2}a - u$, thus $\dfrac{c^2}{a} - (\frac{1}{2}a - u) - \dfrac{c^2}{a^2} - (\frac{1}{2}a - u)^2$ or $\frac{1}{4}c^2 - \dfrac{c^2u^2}{a^2} = y^2$ is an equation for the ellipse, calculating the abscissa from the center.

369. Corollary. In this equation, an opposite but otherwise equal u will give the same y. Thus, if CQ is assumed $= CP$, then QN will $= PM$, $Qn = Pm$, and Dd will also divide the ellipse into similar halves $DMAmd$ and $DNBnd$.

370. Corollary. The quadrant DMA coincides with quadrants dmA (*364*) and DNB, which latter coincides with quadrant dnB (*364*), thus are all four quadrants equal and similar.

371. Corollary. From (*368*) $\frac{1}{4}a^2 - \dfrac{a^2y^2}{c^2}$ becomes $= u^2$. Now let $CE = MP = y$, therefore $EM = CP = u$, and the equation between CE and EM thus has the same form as that between CP and PM. There the abscissa CE was taken on the minor axis, and here the abcsissa CP is taken off the major axis, and an equation becomes out of the other, if only the axes and coordinates are transposed. [??] It must thus be assumed, that in equation (*368*), a is to signify the major axis, and c the minor, if both are not equal.

372. Corollary. If they are equal, then $y^2 = \frac{1}{4}c^2 - u^2$ or the ellipse becomes a circle, whose radius $= c = a = b$ (*366*).

373. Corollary. Every ordinate y is smaller than CD and the further an ordinate is from the center, the smaller it is (*368*).

374. Corollary. If PM; QN; are two arbitrary [undetermined? *unbestimmte*] ordinates, thus $PM^2 = b \cdot \left(AP - \dfrac{AP^2}{a}\right) \cdot QN^2 = b \cdot \left(AQ - \dfrac{AQ^2}{a}\right)$ (*362*). Therefore, $PM^2 : QN^2 = AP \cdot (a - AP) : AQ \cdot (a - AQ) = AP \cdot PB : AQ \cdot QB$; the square of the ordinate varies as the rectangle under the segment of the axis.

375. Corollary. y^2 is $= 0$ for $u = \pm\frac{1}{2}a$ (*363*) and negative for $u > \frac{1}{2}a$; therefore there is no ordinate for abscissas larger than $\frac{1}{2}a$, on both sides of the center, so that the entire ellipse is thus contained in the figure $AMDBdmA$.

376. Problem. To find the focus, F (*358*), of the ellipse.

Solution. From (*368*), $\frac{1}{4}c^2 - \dfrac{c^2CF^2}{a^2} = \frac{1}{4}b^2 = \dfrac{c^4}{4a^2}$ (*366*), thus $CF^2 = \frac{1}{4}a^2 - \frac{1}{4}c^2$ and $CF = \pm\sqrt{(\frac{1}{4}a^4 - \frac{1}{4}c^2)} = \dfrac{\sqrt{(a^2-c^2)}}{2}$ which indicates two foci F, f, at equal distances on opposite sides of the center, as must be the case according to (*369*). I will call $CF = Cf = e$.

377. Corollary. Because $CD = \frac{1}{2}c$, thus $FD = fD = \frac{1}{2}a$.

378. Problem. To find the sum of the lines FM and fM from both foci to a point on the ellipse.

Solution. $FM^2 = (e - u)^2 + y^2 = \frac{1}{4}a^2 - \frac{1}{4}c^2 - 2eu + u^2 + \frac{1}{4}c^2 - \dfrac{c^2u^2}{a^2}$ (*376; 368*) $= \frac{1}{4}a^2 - u\sqrt{(a^2-c^2)} + \dfrac{a^2-c^2}{a^2}u^2$. This is the square of $\pm\frac{1}{2}a \mp \dfrac{u\sqrt{(a^2-c^2)}}{a}$. Now, a positive value is undoubtedly sought for FM. Thus, the symbol must be used for which the root of the square is positive, whatever is placed instead of u, since u should be able to denote every abscissa. Now, the greatest value which u can have is $\frac{1}{2}a$ (*375*). This gives, with the upper sign used, the positive root $\frac{1}{2}a - \dfrac{\sqrt{(a^2-c^2)}}{2}$, and therefore each smaller u gives this root as positive with the use of the above sign. On the contrary, every u that is not larger than $\frac{1}{2}a$ gives, with the use of the lower sign, the negative root. Therefore, $FM = \frac{1}{2}a - \dfrac{u\sqrt{(a^2-c^2)}}{a}$. Further, $fM^2 = (e+u)^2 + y^2 = \frac{1}{4}a^2 - \frac{1}{4}c^2 + 2eu + u^2 + \frac{1}{4}c^2 - \dfrac{c^2u^2}{a^2} = \frac{1}{4}a^2 + u\sqrt{(a^2-c^2)} + \dfrac{a^2-c^2}{a^2}u^2$ therefore $fM = \frac{1}{2}a + u\sqrt{(a^2-c^2)}$. Consequently $FM \pm fM = a$; or, the sum of these lines is equal to the major axis.

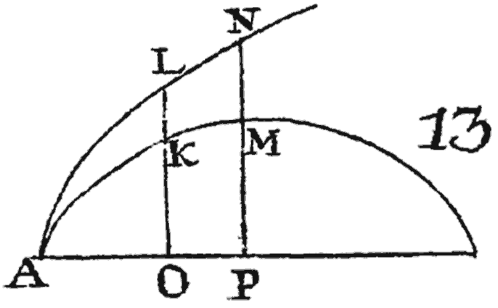
379. Corollary. With an arbitrary piece, z, of the major axis, a circle is described around [*aus*] F, and another around f using $a - z$. Thus, two points on the ellipse are given, and in such a way all of its points can be found.

380. Corollary. If a thread, whose length is a, is fastened with both ends at F and f, and then is extended through every position FMf with a pencil, the pencil will therefore describe an ellipse (*Astronomy. 236.*).

381. Corollary. The closer F and f come together, or approach C, the smaller e is (*376*) and $fM - FM = \frac{2u\sqrt{(a^2-c^2)}}{a}$ (*378*) $= \frac{4ue}{a}$ and thus the ellipse becomes closer and closer to a circle, whose radius is $= \frac{1}{2}a = \frac{1}{2}(FM + fM)$ and whose center is C. For $e = 0$, it is completely transformed into a circle. cf. (*Astronomy. 235. 238.*)

382. Corollary. The distance of the focus from the extremum A is $\frac{a \mp \sqrt{(a^2-ab)}}{2}$ (*378; 366*) where the upper sign is taken for AF and the lower for Af.

Comparison of the Ellipse with the Parabola



383. Theorem. AM, and AN, (*Fig. 13*), are arcs of an ellipse and of a parabola, which share an extremum and each have b as a parameter. One abscissa, x, belongs to both arcs. The ordinates $PM = y$ and $PN = z$ differ less the larger the axis of the ellipse is, if everything else remains unchanged.

Proof. $z^2 = bx$ (*349*) and $y^2 = bx - \frac{bx^2}{a}$ (*362*). Thus, $z^2 - y^2 = bx^2 : a$, which difference of squares becomes smaller, the bigger a is. Also, $x - y = \frac{bx^2}{a(x+y)}$. But $z + y < 2z$ because $y < z$. Therefore $z + y < 2\sqrt{bx}$ and $z - y < \frac{bx^2}{a \cdot 2\sqrt{bx}}$ or $z - y < \frac{x\sqrt{bx}}{2a}$, where the difference between the ordinates shrinks as a grows.

384. Corollary. As well, this difference can never completely disappear. It can only, as a grows, become larger than every given magnitude. The arc of the ellipse can thus, by the increase of a, come closer and closer, and therefore as close as is desired, to that of the parabola. However, they will never become exactly equal. This is expressed thus: the arc of the parabola is equal to the arc of an ellipse whose axis were infinite. It is clear, that for smaller abscissas AO, with the same ellipse [*bei eben der Ellipse*] the difference, KL, of the ordinates, will be still smaller than MN, because $z - y$ vanishes when x vanishes, if b and a remain unchanged. This observation explains (*Astronomy. 306.*).

On the Similarity of Ellipses

385. Definition. If the major and minor axis of an ellipse, and its parameter are α, γ, and β [respectively], the ellipse is called similar (*specie eadem*) insofar as $a : c = \alpha : \gamma$, or $a : \alpha = c : \gamma$ hence, since $b : \beta = \frac{c^2}{a} : \frac{\gamma^2}{\alpha} = \frac{c^2}{a} : \frac{\alpha^2 c^2}{a^2 \alpha}$ or also $b : a = \beta : \alpha$.

386. Corollary. If the abscissa of this ellipse is called t and its ordinate z; we would have $z^2 = \frac{1}{4}\gamma^2 - \frac{\gamma^2 t^2}{\alpha^2} = \frac{\alpha^2 c^2}{4a^2} - \frac{c^2}{a^2}t^2$. Now, we would also have $a : \alpha = u : t$ thus $z^2 = \frac{\alpha^2 c^2}{4a^2} - \frac{c^2}{a^2} \cdot \frac{\alpha^2 u^2}{a^2} = \frac{\alpha^2 c^2}{a^2} \cdot \left(\frac{1}{4} - \frac{u^2}{a^2}\right) = \frac{\alpha^2 c^2}{a^2} \cdot \frac{y^2}{c^2}$ therefore $z = \frac{\alpha y}{a}$ or, if the abscissa has a given ratio with the axis, the ordinate has the same.

387. Corollary. Similar ellipses differ only as to size. All constant lines in them are proportional to one another, and for abscissas which relate to each other as a pair of homologous [*zusammengehörigen*] lines in both ellipses, e.g. as the major axes or the parameter, the ordinates are also proportional.

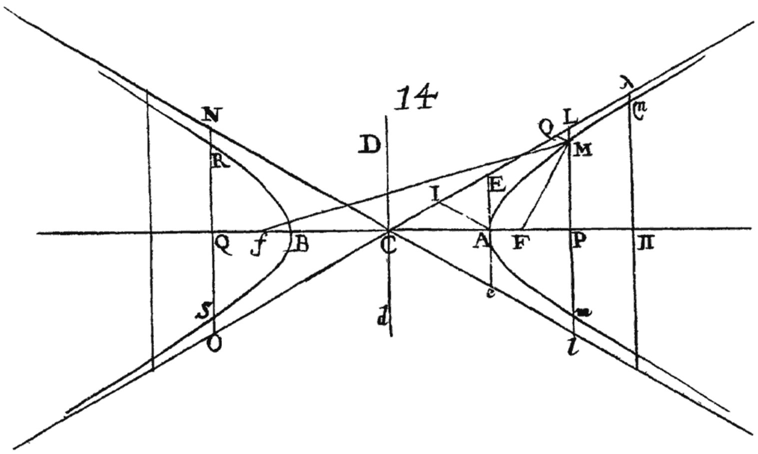
388. Scholium. Herein lies the general concept of the similarity of curved lines, to glean more of which, EULER's *Introdution to Analysis of the Infinite.* Book II, Chapter 18. must be read.

389[a]. Scholium. All parabolas are similar to one another, since if for two parabolas $y^2 = bx$; $z^2 = \beta u$ and $u : x = \beta : b$ thus $z^2 = \frac{\beta^2 x}{b} = \frac{\beta^2 y^2}{b^2}$ or $z : y = \beta : b$. This applies to all lines whose equation only contains one constant magnitude.

The Hyperbola

389[b]. Definition. The **hyperbola** is a curved line, where, for retangular coordinates, $y^2 = bx + \frac{bx^2}{a}$; the lines b and a are again called **parameter** and **axis**, the latter also *latus transversum.*

390. Corollary. If $-a$ instead of $+a$ is placed in the equation for an ellipse (*362*) it will become the equation for a hyperbola.



391. Corollary. In this sense the hyperbola can be considered as if it were an ellipse, whose axis were negative.

392. Corollary. For $y =$, we have $x = 0$ or $= -a$. Thus, if A (*Fig. 14*) is the beginning of the abscissa, and at the same time an extremum of the hyperbola (*363*), and AP is a positive abscissa, another extremum, B, would therefore fall on the opposite side, where $AB = a$ and is considered negative if it is compared with the positive abscissa.

393. Corollary. To every abscissa belong two opposite, but otherwise equal ordinates. The axis thus divides the hyperbola in similar halves.

394. Corollary. For every positive x which is larger than 0, there is a possible ordinate which grows constantly as x grows. Thus, the hyperbola extends two infinite arms along the axis which constatly separate ever further from one another, like the parabola (*355*).

395. Corollary. Because $y^2 = bx \cdot \left(1 + \frac{x}{a}\right) = \frac{bx}{a} \cdot (a + x)$, the square of the ordinate is therefore negative for every negative x which falls between 0 and $-a$, since the factor $\frac{bx}{a}$ is negative for every negative x. However, for an x of that sort [*von den angegebenen*], $a + x$ is positive. Therefore the ordinates are impossible for these abscissas, that is, there is no point of the hyperbola, whose ordinates run through the space A to B.

396. Corollary. However, when a negative x becomes larger than a, $a + x$ also becomes negative, therefore $\frac{bx}{a} \cdot (a+x)$ is positive, being the product of two negative magnitudes, and the ordinate becomes possible again. There are thus possible ordinates for such negative abscissas, that is, they belong to points on the hyperbola.

397. Corollary. These ordinates grow constantly, if the negative x grows. Since in these cases $a+x$ becomes an ever larger negative quantity, and $\frac{bx}{a}$ likewise. Therefore the product becomes a larger positive quantity.

398. Corollary. The hyperbola also extends infinitely out from B with two arms, BR and Br, which open ever wider as in (*394*).

399. Corollary. If $\sqrt{ab}=c$ is the minor axis of the hyperbola, as in (*361*) (*latus rectum, 389*) thus $y^2 = \frac{c^2}{a}\cdot x + \frac{c^2x^2}{a^2}$.

400. Definition. If $AC = \frac{1}{2}AB$, then C is the center of the hyperbola.

401. Corollary. Here there is no ordinate through the center (*395*) as with the ellipse (*365*). That is to say, the hyperbola were an ellipse, whose minor axis were impossible (*391*).

402. Problem. To find an equation for the hyperbola, if the abscissas $CP=u$ are calculated from the center.

Solution. Because $x = u - \frac{1}{2}a$, we have from (*399*), $y^2 = \frac{c^2u^2}{a^2} - \frac{1}{4}c^2$.

403. Scholium. This equation can be obtained from (*368*) if $c\sqrt{-1}$ is written instead of what is there c (*401*).

404. Corollary. The perpendicular Dd divides the hyperbola into similar halves, as in (*369*). These are called opposite hyperbolae although together they actually form one single hyperbola. Also, in virtue of (*393*) and (*404*), equal and similar parts of the hyperbola lie within the right angles DCA, ACd, dCB, and BCD, as in (*370*).

405. Corollary. For the focus F (*376*), the square of the ordinate is $= \frac{1}{4}b^2 = \frac{c^4}{4a^2}$ (*399*) $= \frac{c^2u^2}{a^2} - \frac{1}{4}c^2$ (*402*) thus, if u here signifies the abscisse for the focus, $u = \pm\frac{\sqrt{(a^2+c^2)}}{2}$ for both lengths CF, and Cf. Also $AF = \frac{\sqrt{(a^2+c^2)}-a}{2}$.

406. Corollary. $FM^2 = \left(u - \frac{\sqrt{(a^2+c^2)}}{2}\right)^2 + yy = u^2 - u\sqrt{(a^2+c^2)} + \frac{a^2+c^2}{4} + \frac{u^2c^2}{a^2} - \frac{1}{4}c^2 = \frac{a^2+c^2}{a^2}u^2 - u\sqrt{(a^2+c^2)} + \frac{1}{4}a^2$ thus $FM = \frac{u\sqrt{(a^2+c^2)}}{a} -$

$\frac{1}{2}a$ if it is to be positive, as in (*378*). Here, we always have $u > \frac{1}{2}a$ (*395*). Similarly [*eben so*], $Fm^2 = \left(u + \frac{\sqrt{(a^2+c^2)}}{2}\right)^2 + y^2$ and $Fm = \frac{u\sqrt{(a^2+c^2)}}{a} + \frac{1}{2}a$. Therefore $Fm - FM = a$, or, the difference between the two lines is equal to the major axis, just as the sum was with the ellipse (*378*).

407. Scholium. From this can be derived a means to draw the hyperbola, as with the ellipse (*379*). Also, tools to describe these three curved lines with one continuous motion are founded on the properties of the focus (*360, 378, 405*).

408. Corollary. For the straight line CL, let $LP = z$ and $z = \frac{cu}{a}$, thus $AE = \frac{1}{2}$ (*400*) thus (*402*) $z^2 - y^2$, or, $(z+y)\cdot(z-y) = \frac{1}{4}c^2$ thus $z - y$ or $LM = \frac{cc}{4\cdot(z+y)}$ which diminishes constantly if u grows, and can become smaller than any given magnitude, although it can never become zero so long as u and therefore z and y are determined [*bestimmte*] magnitudes. Thus LM, the distance of arm AM of the hyperbola from the straight line becomes ever smaller and smaller, and tinier than any given magnitude, however it will always remain something. The hyperbola approaches this straight line ever closer and closer, but never reaches it. This straight line is called the **asymptote**.

409. Corollary. Its portion CL pertains to the arm AM, its portion CO to the arm BS. Similarly [eben so] there is another asymptote Nl, whose equation is $QN = -\frac{cu}{a}$. Its portions CN, Cl, pertain to the arms BR, Am.

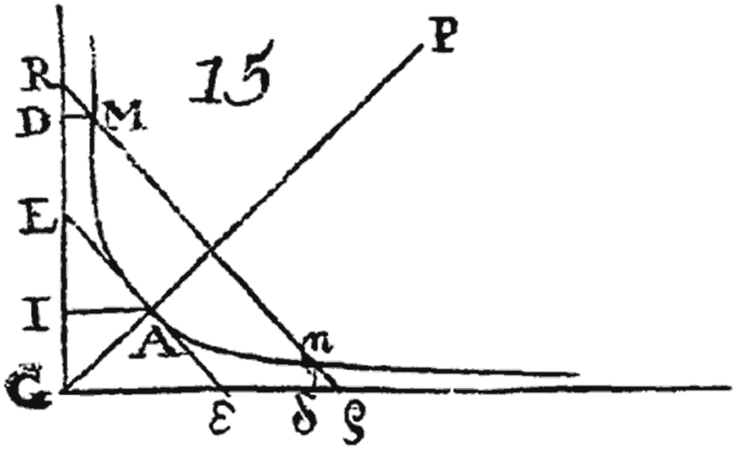
410. Corollary. Because $Ml = y + z$, we thus have constantly $LM \cdot Ml = \frac{1}{4}c^2$ (*405*).

411. Corollary. Let QM be parallel to the asymptote Cl. Then $QML = ClL = CLl$ (because $PL = Pl$ and form a right angle at P) thus $QM = QL$, further $CE^2 = CA^2 + AE^2$ thus (*408*) $CE = \frac{\sqrt{(a^2+c^2)}}{2}$ and $Ee = c$. Now $CQ : Ml = LQ : LM = EC : Ee$ thus $CQ = \frac{(z+y)\cdot\sqrt{(a^2+c^2)}}{2c}$. Also $Ee : eC = LM : MQ$, or, $MQ = \frac{c^2}{4(z+y)} \cdot \frac{\sqrt{(a^2+c^2)}}{2c}$. Therefore $CQ \cdot QM = \frac{a^2+c^2}{16}$.

412. Corollary. CQ and QM can be considered as coordinates [taken] on [*auf*] the asymptote. If the former $= p$ and the latter $= q$, then $pq = \frac{a^2+c^2}{16}$ is the equation of the hyperbola between the asymptotes [in terms of the asymptotes?].

413. Corollary. If AI is also parallel to Cl, then $CAI = ACe = ACE$; thus $CI = AI$; and $CI \cdot AI = CI^2$. Hence $CI^2 = \frac{a^2 + c^2}{16}$ (*412*). This square is called the **power of the hyperbola**.

414. Problem. To find the ratio of MR and $\mu\rho$, if an arbitrarily drawn straight line, (*Fig. 15*), cuts the asymptote in R, ρ and the hyperbola in M, μ.



Solution. MD and $\mu\delta$ are drawn parallel to the asymptotes $C\rho$, CR. Thus

$$MR : MD = R\rho : C\rho$$
$$R\rho : \mu R = C\rho : C\delta; \text{ therefore,}$$
$$MR = \frac{MD \cdot \mu R}{C\delta}$$

and, similarly, overlooking for the moment the manner in which the Greek and Latin letters are interchanged, $\mu\rho = \frac{\mu\delta \cdot M\rho}{CD}$, thus $MR : \mu\rho = CD \cdot MD \cdot \mu R : C\delta \cdot \mu\delta \cdot M\rho = \mu R : M\rho$ (*411*) $= M\mu + MR : M\mu + \mu\rho$ therefore $\mu R : M\rho = M\mu : M\mu$ (*Arithmetic. Chapter V. Lemma 34. II.*) or $\mu R = M\rho$.

415. Corollary. If the position of the asymptote is given with a point M, all of the remaining points are found if all lines $MR\rho$ are drawn, and $\rho\mu$ is always $= RM$. This also occurs, if the points M and μ are in opposite hyperbolae (*404*), as can be easily seen by sketching the appropriate figure.

416. Corollary. From the given things (*415*), the axes are so given: MD is drawn parallel with $C\rho$ and the angle $RC\rho$ is halved by CP. CI is taken $= \sqrt{CD \cdot MD}$ and therefore $= \frac{\sqrt{(a^2 + c^2)}}{4}$ (*412*) and IA is drawn parallel to $C\rho$, so that CA is the semi major axis, and AE, perpendicular to it, equal to the semi-minor (*413*).

417. Definition. The **equilateral hyperbola** is that, whose axes (*389, 399*) are equal to each other and to the parameter.

418. Corollary. Therefore, in the equilateral hyperbola $CA = AE$, and it follows that $ACE = \frac{1}{2}R$ and the angle of the asymptotes ECe is a right angle. Hence, this hyperbola is also called rectangular. Its equation is $y^2 = u^2 - \frac{1}{4}a$ (*402*).