Anfangsgründe
Part 3 Section 2
Analysis of the Infinite

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1 Preface

Secrets must [*doch*] be something very lovely for the human understanding, because they have always been sought out in any given field of inquiry, for thousands of years, and made famous because of their distinctiveness and persuasive qualities. Once see without my bringing to your attention, that I'm speaking of the study of the infinite within mathematics, which has been approached and advanced upon[?] by many authors, even those who have furthered the extent of knowledge. Magnitudes which are different but looked upon as equal; infinitely large things which become nothing when compared to others, and infinitely small [things] which become infinitely large when compared; curved lines, which consist of straight parts, whose straightness one can bend again at will if one wishes, and countless other such propositions which touch upon the origins shake the foundations [*wieder die ersten Gründe anstossen*] of all human understanding should be looked at by students of mathematics as the most elevated and awe inspiring task. But could he hope to advance upon his admirable predecessors, if he, as Haller said: "fills with smoke the shimmering vault?"

The older thinkers [*Die Alten*], in order to avoid the use of the term "infinite," and all that might have some relationship therewith, applied a diligence which our indulgence [*Geduld*] transgresses, and even Descartes trembled when he used the word "unbounded" [*indefinitum*] instead. The word itself becomes unimportant, if one can determine a clear understanding of its meaning. Just as I determined in the beginning of these current works, everything seems clear to me, and the secrets of the infinite, which have seemed to be out of the grasp of some authors, because our finite being can not touch the infinite, are transformed into very obvious statements [theorems?], partly into a play on words.

But they are no longer a mere play on words when you look at them as expressions [?], which express something concisely and profoundly, which usually requires an extensive explanation. What differentiates the bold/dashing expressions of a composer, from those of perhaps the logical, but dry/dull dictions of a philosopher, is approximately the same difference as the calculation of the infinite compared to the proofs/demonstrations of old: Even the solid earth itself began [Also has the solid ground learned it first from a spirit] as an idea [thought-object], in which the wits of the poet, the insight of the philosopher, the rigor of the geometer and the comprehensiveness [*belesenheit*] of the universal historian unite. In the first experiment that he produced from his findings through the calculation of the infinite, in his quadrature of the circle,[1] he already showed how we, unhindered by our limited powers, could grasp the infinite. We do not conceive of all of its parts, but we are familiar with the law, that all these parts observe in common. We will never know all terms of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$ but that each pair of subsequent terms must be $+\frac{1}{4n+1} - \frac{1}{4n+3}$, we grasp completely.

By means of the Newtonian Parallelogram we find, for a given equation, the first terms of an infinite series, and the principle of its continuation, through which all further terms are determined; it is not clear to us how they are composed [*beschaffen*], when we have established the first; For first we could choose from a variety [*andere und andere*], and therefore could produce a number [*andere und andere*] of series: Is it something as absurd, as the free thinker, and the too soft-hearted Fromme tried to persuade us, that our fate for all eternity, is determined by our current lifetime? Could our present years simply be the first terms in the parts of an equation series,[2] whereby all the ensuing will be determined without end?

Paucula iam series mostrat primordia nascens
Lege sua partes haec sine sine regunt:
Qualiter a vita quam bis sex lustra coercent
Aeua tenent formam non numeranda suam.

As soon as we leave the idea of the continuing infinite parts, to try to comprehend the parts themselves, as soon as we start to view the infinite as something real or existing, we overturn our analysis of the real things [*Dinges*]. The power/ability to determine anything lies in ones being/essence, but to be **determinant** means, to let yourself be specified [*angeben* ?], and therefore necessarily finite. This is pointed out by Barrow,[3] from whose lectures on geometry Newton certainly, and perhaps Leibniz as well, has derived his first notions of the calculation of the infinite. On the other hand, the infinitely small of large magnitudes, **by their nature**, can not be determined as Hausen expressed,[4] who never forgot to remind his audience that to say something happens at infinity, is to say it never happens, and the location where one parallel meets with another, the hyperbola with the asymptote, is nowhere.

[1] Acta Eruditorum Leipsic. February 1682

[2] Algebra. 649.

[3] *desultoriam & indeterminatam quantitatem habere, est havere nullam; est enim aliquid determinate, quicquid est, quod vbique est, nusquam est.* BARROW *lect. Mathem. anni 1666. quinta p. 262. ed Londin 1684*

[4] *natura sua indeterminabilis.* HAUSEN *El. Ar. Schol. Prop. 20.*

In the discourse in which Galileo first taught the Law of Falling [*Falles*], and began his investigation into the solidity [*festigkeit*] of material bodies/objects,[5], one finds many odd [*seltsam*] sounding sentences about the infinite[6] and he leaves his readers in all the confusion [*verwirrung*], in which they can be brought through his sophistry: How I believe out of a philosophical wantonness *spitz-*

[5] *Disorsi e dimostrazioni matematiche intorno a due nuoue seienze &c.* (*ed. Elsey. 1638; 4to*) *dialogo 1.*

[6] **UNTRANSLATED.** Ich will einen davon beybringen, den Stedler (Fränkische *Acta. Erud. & Curiosa X. Samml. VII. Art.*) vorgetragen hat, doch ohne einmal das Buch des Galiläus anzuzeigen, darinnen er sich findet. "Jedes Kreises Umfang soll seinem Mittelpuncte gleich seyn." Ich will bey meinem Vortrage zum voraus setzen, dass man den 66 Satz meiner Geometrie, und die dazu gehörige 127 Fig. vor Augen habe. Ein Kreis der QH zum Halbmesser hat, ist so gross als der Ring zwischen den beyden Kreisen die QE und QL zu halbmessern haben. Aber wenn QL in KG fällt, so verwandelt sich der Kreis in den Mittelpunct K, weil $QH = 0$; und der Ring in dem Umfang eines Kreises mit KG; weil $QE = QL$ also ist der Mittelpunckt den Umfange gleich. So schliesst *Salviati* beym Galiläus (a.a.O. 28. S.) Aber aus $QH^2 = QL^2 - QE^2$ oder $QH^2.\pi = (QL^2 - QE^2).\pi$ folgt für $QE = QL$ nichts als $QH = 0$ und $0 = 0$, ob sich also wohl die Gleichheit dex Kreises und des Ringes bis auf den Augenblick da beyde verschwinden erhält, so ist es doch nicht erlaubt den Mittle punct K als den letzten unter den immer abnehmenden halbmessern QH; und den Umfang eines mit KG beschriebenen Kreises als den letzten unter den immer abnehmenden Ringen zu betrachten. Wenn sich eine Grösse durch beständige Abnahme in eine andere verwandelt, so muss ohne Zweifel der Unterschied zwischen beyden immer kleiner und kleiner werden. Aber man kann sich keinen Unterschied und folglich auch keinen immer abnehmenden Unterschied zwischen K und QH; oder zwischen dem Umfange und dem Ringe vorstellen.

Wenn es erlaubt ist, von der beständigen Verhältniss zwoer veränderlichen Grössen, auf die Verhältniss dessen zuschliessen, das noch bleibt, wenn die Grössen nicht mehr sind, so will ich mit leichter Mühe noch grössere Ungeheuer, z. E. einen Punct der sich selbst nicht gleich ist, zum Vorscheine bringen. In der ersten der zu gegenwärtigem Werke gehörigen Figuren, sey TM eine gerade Linie die mit TP einen Winkel von 30 Gr. macht: So ist allemahl $MP = \frac{1}{2}TM$ wo auch M genommen wird; Man lasse also MP immer näher an T rücken, bis sie endlich in T fällt, so verwandelt sich MP sowohl als TM in T; und es ist also der Punct T als die letzte MP betrachtet, halb so gross als vollkommen eben derselbe Punct T, als die letzte TM betrachtet.

Ich wollte dass ich hiebey nicht erinnern müsste, dabb man aus der Gleichung $2.0 = 1.0$ nicht etwa schliessen soll: ein Nichts könne halb so gross als ein anderes Nichts, oder $0 : 0 = 1 : 2$ seyn. Die Gleichung sagt soviel ich einsehe nur soviel: dass Nichts undNichts dazu auch noch Nichts ist. Aus $2.y = 1.x$ folgt $y : x = 1 : 2$; wenn man aber hier 0 statt y und x setzt, so bedeutet 2.0 nicht so etwas wie $2y$ bedeutet. Dieses heisst ein Ding zweymahl nehmen, und jenes, Nichts zweymahl nehmen. Wollte man sich verstatten ein Nichts hier statt Nichts zu setzen, und das Nichts wie ein Ding zu betrachten das man zweymahl nähme, so würde man sich den Erinnerungen aussetzen die Herrn Premontval in den *Mem. de l'Ac. R. de Prusse 1754; 421. und 433 S.* wieder den Wolfischen Beweis vom Satze des zureichenden Grundes gemacht hat. Oder, um die Sache weniger metaphysisch vorzutragen: Man würde den Grund der erhabensten Rechnung, mit dem Witze eines alten deutschen Formenschneiders Johann Daniel Müllers legen, der nach der Mitte des vorigen Jahrhunderts gelebt hat. Ich besitze von ihm den Niemand nach dem Leben abgebildet, und in schönen duetschen Versen alles zu seinem Lobe und Tadel gesagt, was Niemand thut, oder gethan haben will. Bey diesem Niemande, und bey den zwey Nichtsen deren eines noch einmahl so gross als das andere ist, fällt mir ein lateinisches Heldengedicht ein, damit ich diese lange Anmerkung schliessen will

Nullus & Nemo
Mordebant se in sacco:
Nullus clamabat
Nemo audiebat.

fundigkeiten? Otherwise he would have no-doubtedly resolved these difficulties, as he encountered them: this becomes evident through the proper recollection [*errinerung*] on the 36th page of his works, that the question, whether a line contains an infinite amount of parts is answered: It contains any given number of parts.

The more elderly Sturm also collected the incomprehensible teachings of mathematics in his own treatise.[7] They are all based/founded on the Infinite, and become inconceivable only through their delivery. The equality of parallelograms of a baseline [*grundlinie*] between the same parallels also comes up here. There I really would not have sought here: But since I found them nonetheless, therefore I endeavored to find another similarly great incomprehensibility [*unbegreiflichkeit*], and that I did not find: How it was the Ox Skin of Dido, cut into a strap, gave a circumference into which Carthage could fit.[8]

P. Boscowich has a new collection of the secrets, containing his own nonsense/absurdities about the infinite.[9] But in that you can only find stead/ground, if you view the infinite as something physically/substantially existent, and to accept Boscowich's expression in "infinitum absolutum." If one takes only magnitudes which can either infinitely increase or decrease, as indeterminate, then everything can be explained without difficulty, as he also reminds. That the use of the infinite would create so many objections was clear from the outset, and makes reference to Fontenelle's thoughts[10] on this, while the geometry from the aspect where the infinite appears/arises, with natural philosophy, upon which the inner being of the body depends, is so unknown to us. This notion has, like so many others the author of this book, more humor than profundity, if indeed true humor can be without grounds. I had to be very disappointed, if he were not to draw considerably upon an instance of Germans sagacity and trueness. Fontenelle takes magnitudes, which fall between the infinite and the finite; it is these, which we express by $\infty^{\frac{1}{2}}$ and the like; through these he claims the transition from the finite to the infinite takes place. But by these bridges/pathways which lead from the finite to the infinite, he did not take into consideration, like Haused joked, that more of those bridges would be necessary in order to get out of the finite.

That Geometry should inherit the infinite from Natural Philosophy, is so wrong, that rather Geometry, by conclusions, had wished to bring the infinite into Natural Philosophy, of which I have already opened my thoughts on elsewhere.[11] With these conclusions, the philosophers of old have become so perplexed/entangled [*verwirrt*], that Fromond wrote a book in which he col-

[7] *De Matheseos incomprehensibilibus. v. Io. Christoph Sturmii praelect. Acad. ed. a Dan. Algoeweto 1722.*

[8] Translator's Note: From Virgil's *Aeneid*, Book 1. "*Devenere locos, ubi nunc cernes ingentia moenia, surgentemque arcem novae Carthaginis; mercatique [-sunt] solum, quantum possent circumdare taurino tergo, Byrsam de nomine facti.*"

[9] *De transformatione locorum geometricorum; §. 880. sequ.* am Ende seiner Anfangsgründe der Kegelschnitte die den dritten Theil seiner *Elementorum Matheseos (Rom 1754.)* ausmachen.

[10] *Elemens de la geometrie de l'infini; pref.*

[11] *Anfangsgründe. Geometry. Definition 3. Scholium.*

lected/gathered many comparisons [*vergleichen*], to which he gave the name of a "Labyrinth."[12] Blancan, having a chance [*gelegenheit*] text of Aristotle from uninterrupted lines, accounted[13] that the investigation of whether infinite beings/substances, and how they exist [*vorhanden*] in nature, must follow from other fundamentals and not geometry: an account which some disputers of the monad should have read, before they were allowed to print false conclusions for geometric proofs.

Maclaurin, in his "Treatise on Fluxions," diffused/scattered all haze about the imagined secrets of the infinite. We could parody his book and give it a very notorious/infamous title, and call it "Infinity not Mysterious." Now the synthetic method, which he uses in the style of an Englishman, brings forth two underlying inconveniences. The first, the big explanation, he is guilty of himself that one by those tricks [*kunstgriffen*] which so usefully shorten the most profound investigations, not to be too detailed/lengthy in ones delivery of reasons. Many of his proofs, which, for example, provide various cases for a single theorem, are so similar to one another, that one only need read one to get an overview of them all. But in a book which teaches us a superb portion of the mathematical method, it would have been useful/appropriate to direct the presentation of the foundations in such a way that one could see how they were derived.

In my prevailing works I mostly occupied myself with the first or last portion, which Newton used in his "Principia." Without care I had determined that, just as the habitual phraseology allows the justifying of the infinite, so have I still used them myself where it occurred without all-too-large run-ons, or of course always so used, that one sees how they are only shortcuts of the discourse. A woman, which nature fashioned well, sometimes displays herself through a dress after the custom of a distorted appearance: and the analysis of the infinite seems to have, when veiled/cloaked in such false expressions, which thereby are not covered, that they offer the informed/rational mathematician inexhaustible estimates/assessments of the findings/fabrications: then for the eyes of a soul/spirit to whom only truth is beautiful, this would be an absolutely hideous realm. But luckily these are only errors in her dress; sometimes incompetent tailors served her, sometimes she intentionally and with haste threw a sloppy dress [*Salope*] around herself; and the geometer can become so enchanted, as Ovid said:

> *Vt stetit ante oculos posito velamine nostros*
> *In tot nusquam corpore menda fuit.*

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[12] *Liberti Fromondi Labyrinthus, s. de compositione continui liber vnus, philosophis, mathematicis, theologis vtilis & iucundus. Antu. 1631.*

[13] *Blancani loca Aristotelis mathematica p. 209.*