

Anfangsgründe

Volume I Section 1

Plane Trigonometry

Abraham Gotthelf Kästner.

1761

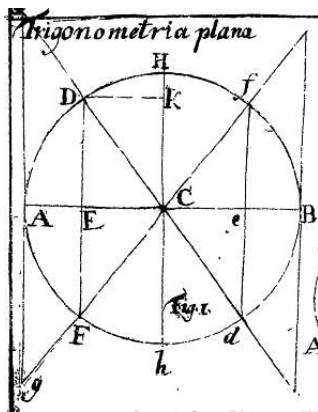
Translated from the German by Sky Shields

1. Definition.

Plane trigonometry teaches how, through calculation, to find the unknown sides or angles of a triangle, from several given sides and angles, if such are given in numbers.

Corollary.

Among the given things must be at least one side. The angles alone determine innumerable similar triangles: (speciem trianguli) (Geometry, theorem 26.)



2. Definition.

To both endpoints of an arc AD , radii are drawn from C (fig.1). On the one radius, CA , falls a perpendicular, DE , from the endpoint of the other; this line is called the **sine** of arc AD .

1. Corollary. If one extends this line until it cuts the circle again at F , then $EF = DE$, (Geom., theo. 17.) and the arc $DA = AF$ (Geom., theo. 22, cor. 2) The sine of an arc is thus the half chord of the double arc.

Example. If the arc $DAF = 60^\circ$, the chord DF thus = CD (Geom., theo. 23, cor. 5), and DE , $\sin DA = \sin 30^\circ$, or, the sine of 30 degrees is equal to half the radius. If $DAF = 90$ degrees, then $DF = \sqrt{2(CD \text{ squared})}$ (G. 15. S.)

thus $DE = \sin 45\text{deg.} = \frac{\sqrt{2(CD \text{ squared})}}{2}$.

2. Corollary. For arcs DHB which are over 90° , the sine is also DE ; and any two arcs, which together make up 180 deg., have exactly the same sine.

If one considers the arc $ADHf = BHD = 180^\circ - AD$, so will Bf be $= AD$, and the perpendicular fe would be sine of the angles Af and Bf . And because $ACD = BCf$ (Geom., theo. 22), fe is thus $= DE$ and $Ce = CE$ (Geom., theo. 3). If $AH = 90^\circ$; then here Hf is $= HD$, or in other words, AD amounts to as much below 90° as Af does over it; thus, angles, regardless of what side of the quadrant they are measured from, have the same sine.

3. Corollary. If the arc AD grows to 90° , D then moves to H , and therefore F to h , the chord DF becomes equal to the diameter Hh (Geom., theo. 18.) The sines of arcs under 90 deg. thus grow with their arc (cor. 1) until, for $AH = 90^\circ$, the sine, HC , becomes equal to the radius. For $Af > 90$ deg., one takes $Ad = Af$, thus the chord fd ever diminishes the greater $Af = Ad$ becomes; the sines of arcs over 90 deg. thus diminish where the angle increases; until for $AHB = AhB = 180^\circ$, chord and sine become $= 0$. This same thing becomes clear from conceiving of the triangle DCE , fCe , as also from cor. 2.

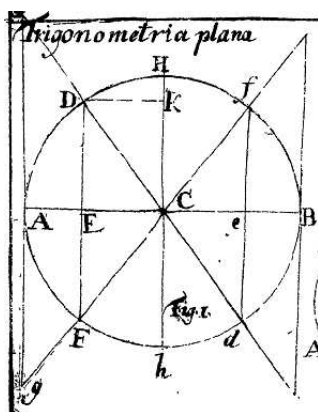
4. Corollary. If DC is extended until it cuts the circumference in d , then de is the sine of arc $AHBd$; An arc over 180° thus has a sine which is as large as the sine of the arc under 180° by which that arc exceeds 180° ; That is, $Bd = AD$. This also holds for the arc $AHBF$ which is over $3 \cdot 90$ or 270° ; and $FE = fe = DE = \sin BHD$ is the same sine as the excess of BhF over 180° , which is $= BHD$.

5. Corollary. The sines of all arcs under 360° are thus smaller than the radius, excepting the sine HC for 90 deg. and the sine hC for 270 deg. They increase from 0 to 90 deg.; diminish again from 90 to 180 deg., where they become $= 0$, grow again from 180° until 270° and shrink from 270° to nothing at 360° .

6. Corollary. One can thus look on them all as actual fractions of the radius, which one therefore calls *sinum totum* (whole sine).

Scholium. If the entire circumference is added one or more times to a given arc, there appears an arc over 360° as in $ADBFAD$; It is clear, that these arcs will have exactly the same sines as above.

7. Corollary. Because the sines between 180 and 360 deg. fall on the negative side of the diameter AB , they are thus negative. (Ar. 1. 90.) This is also clear from the way that they were derived according to Corollary 5. The disappearance [becoming $= 0$] is a change into the negative. (Geom., theo. 12, cor. 14) Let the radius, or total sine be $= r$; $AD = 30^\circ = Bd = AF$; thus $AHBd = 180^\circ + 30^\circ = 210^\circ$ and $AHBF = 360^\circ - 30^\circ = 330^\circ$; and so $DE = \sin 30^\circ = \frac{1}{2}r$, and $de = FE$ or $\sin 210^\circ = \sin 330^\circ = -\frac{1}{2}r$.



The opposite values in the two sines DE , de , is derived from their directions, which are opposite. One may go to each from the point where they cut AB to those where they meet their arc, or from the one to the other. But these directions are not opposite with DE , fe , therefore these sines have the same [negative or positive] sign.

8. Corollary. If one calls AH , HB , Bh , hA , the **first**, **second**, **third**, and **fourth** quadrants, then the sines in the first two are positive, and in the last two, negative.

3. Definition.

DH (cor. 2) is called the **complement** (complementum) of arc AD ; that is¹, it “completes” AD to 90° , and $DK = \sin DH$ and is the **cosine** (sinus complementi) of arc AD .

1. Corollary. Because $DK = CE$, the square of the cosine = r^2 minus the square of the sine; thus the cosine is given when one has the sine. Similarly also for AE , the difference between the radius and the Cosine; which is called the **quersine** (sinus versus; sagitta).

Example. $\cos 30^\circ$ or $\sin 60^\circ = \sqrt{(r^2 - \frac{1}{4}r^2)} = \sqrt{\frac{3}{4}r^2}$.

2. Corollary. $\cos Af = Ce = -CE$ (Ar. 1. 90.) or the cosine of an arc between 90 and 180 deg. is the same as the cosine of the arc by which it falls short of 180 deg., but negative.

4. Definition.

AG , the line touching A , which is cut in G by the extended CD , is called the **tangent** of arc AD and CG its **secant**.

Example. If $AD = 45^\circ$, then $AG = CA$ or $\tan 45^\circ = r$ and CG or $\sec 45^\circ = \sqrt{2r^2}$.

1. Corollary. The tangent of the arc ADf is $Ag = -AG$ for the supposed magnitude of this arc (definition 1, corollary 2); the tangent of arc $ADfd$ is again AG itself; and $Ag = -AG$ the tangent of arc $ADfdF$; or the tangents are in the first and third quadrants positive; in the second and fourth, negative.

2. Corollary. If D moves toward H , G goes farther and farther from A until it reaches the infinite, when D falls on H . (geom, theo. 12, cor. 12) The tangents thus, in the first quadrant, grow with their arcs until $\tan 90^\circ = \infty$ (which symbol indicates the infinite); in the second the tangent decreases as the arc grows; until $\tan 180^\circ = 0$. In the third, positive tangents grow again with their arcs until $\tan 270^\circ = \infty$ and in the fourth, negative tangents decrease until $\tan 360^\circ = 0$. This all coheres with Geometry, section 12, corollary 14.

3. Corollary. If, for brevity, the arc AD is set = x ; then, because $CE : ED = CA : AG$, $\cos x : \sin x = r : \tan x$; or $\tan x = \frac{r \cdot \sin x}{\cos x}$.

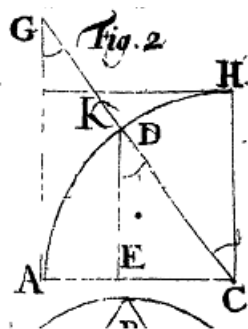
¹ DH heisst des Bogens AD *Ergänzung*; nämlich zu 90 Gr. . . .

4. Corollary. The same also follows from def. 3, cor. 2 and prop. 2, cor. 7 and Arith. 1. 90. etc. as from cor. 2 here. E.g., at H and h , $\cos x = 0$; thus $\tan x = \infty$. That is, a finite magnitude divided by 0 gives an infinite quotient; or, in other words: when the divisor shrinks to become smaller than every magnitude which could be given, the quotient grows and overtakes all magnitudes which could be given.

5. Corollary. $CE : CD = CA : CG$ or $\cos x : r = r : \sec x$; thus are secant and tangent given by the cosine and sine, i.e., through the sine (prop. 3, cor. 1). Also, $\sec x$ is $= \sqrt{r^2 + (\tan x)^2}$ squared.

5. Definition.

The tangent HK (fig. 2) of the complement DH (3. Prop.) is called the **cotangent** of arc AD and CK the **cosecant**.



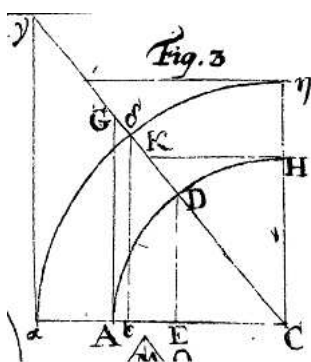
1. Corollary. Because $AGC = GCH$, we have $AG : CA = CH : HK$, or $\cot x = \frac{r^2}{\tan x}$, and $ED : CD = CH : CK$, or $\sin x : r = r : \csc x$, thus the cotangent and cosecant are also given by the sine.

2. Corollary. For another angle y , the $\cot y = \frac{r^2}{\tan y}$, thus $\cot x : \cot y = \tan y : \tan x$.

3. Corollary. The cotangents thus have in their various quadrants the same signs as the tangents.

1. Theorem.

The trigonometric lines (def. 2. 3. 4. 5.) are proportional to the same arcs as the radius.



Proof. With another arbitrary radius $C\alpha$, fig.3, one inscribes the arc $\alpha\delta$, which will be similar to AD , between the sides of the angle ACD , and extends the sine $\delta\epsilon$, cosine $C\epsilon$, tangent $\alpha\gamma$, secant $C\gamma$; thus we have (Geom., theor. 26)

$$CD : C\delta = DE : \delta\epsilon = CE : C\epsilon \text{ and}$$

$$CA : C\alpha = AG : \alpha\gamma = CG : C\gamma \text{ and}$$

$$CH : C\eta = HK : \eta\kappa = CK : C\kappa .$$

1. Corollary. If one divides both radii into an equal amount of equal sized pieces, which simply become larger for the larger radius; there will be as many of these lines for the arc $\alpha\delta$ as there are of the corresponding length for AD .

2. Corollary. The ratio of each of these lines to their corresponding radius comes purely from the size of ACD , which is the measure of arc $\alpha\delta$ as well as AD , if one conceives of both as a quantity of

degrees. (Geom., theor. 22. cor. 6.) One can thus conceive of them as lines which refer to the angle: their actual size then depends on the size of their radii. That is, if CD is assumed to coincide with the side of a given angle, then its sine, DE , is given for this radius; and so another radius, $C\delta$, falling on the same line, gives another sine $\delta\epsilon$. So it is with both radii for each of the remaining [trigonometric] lines which are determined by the sine. (def. 4, cor. 3, 5 and def. 5, cor. 1) One divides the radii, however, which may be big or small, into the same amount of parts, so that each of these [trigonometric] lines becomes a number of such parts, which only depends on the angle and is the same for all radii.

Call each radius 100; then the sine of an angle of 30 deg. is always 50. I.e. if $\alpha C\delta = 30$ deg.; then DE has 50 of the same parts CA has 100 of, $\delta\epsilon$ 50 of the same parts as $C\alpha$ has 100 of.

3. Corollary. Thus, this means that now x expresses the angles in degrees, and the expressions $\sin x$, $\cos x$, $\tan x$, etc. are numbers which belong to each angle, and a radius is assumed at the outset to be divided into a known number of parts.

4. Corollary. The positive and negative signs of these lines change with the angle for obtuse angles, and for angles which amount to over 180 deg., and so on (?) for the arcs which are their measure.

2. Theorem. Proposition.

If the radius is given a definite number of equal parts, find how many of such parts shall be contained, for each angle, in each of the above-mentioned lines.

1. Scholium. That the proposition requires nothing impossible, is clear from Theorem 1, Corollary 2.² However, the methods by which to accomplish this are either too lengthy to find a place here, or they require methods from algebra and the infinitesimal calculus. The former may be found in *Joach. Rhetici* and *Valent. Othonis Opere Palatino de Triangulis*; in *Stevin's Cosmographie I. Theile*, Gellibrand's *Trigonometria Britannica*; *Ozanam Tables des sinus*. Wolf's *El. Trigon. lat. Cap. 1*. Tables are calculated from them which contain the numbers for these lines and their logarithms, for all degrees and minutes of the quadrant, or, in the *Trig. Brit.*, for all degrees and hundredths of a degree. These are adequate for the present goal. For more accurate astronomical calculations, further divisions of the minutes will yet be required. If one has learned the great advantage to be afforded by the use of these tables, one will be able recognize for himself the methods for calculating these tables in the above given or other literature. They rest purely on the general theorems of geometry, many of which, however, could find no place in the following short *Anfangsgruende*. Algebra can serve as an aid with its calculations, and gives prescriptions according to which these tables might be calculated with much greater ease. From this, one can much more easily recognize the possibility of

²1. S. 2. Z.

such tables in general, and prove their values himself, as well as learn to apply them to altogether broader applications, since undertaking the calculations anew were neither a human feat³ nor of any use.

2. Remark. In the calculation of the customary/current tables, the radius has been taken as = 10,000,000,000 [parts]; wherefore its logarithm will be = 10. And in such parts of the radius were all lines, the natural sines [etc.?] and their logarithms, the artificial sines [etc.?] calculated. However in the common issuances⁴ of these tables, the last three digits of all lines so calculated have been thrown away, i.e. they have been given for a radius of only 10,000,000 parts. The logarithms remained unchanged, whose indices are thus all too large for the numbers under which they stand. This gives no error in trigonometric usage, if the calculations are performed using logarithms alone, since then they will be looked on as the logarithms of lines which belong to certain angles, and one need not be concerned how large the lines themselves are. However, if one were to look upon the logarithms as logarithms of the numbers standing below them, one could be led into error. In *Trigon. Bitann.* the radius for the sines is taken as one hundred trillion.⁵ The logarithms however are worked out for the just mentioned radius. Moreover, one looks at the lines which, as was said before, under known circumstances could become negative, as positive, when dealing with their logarithms, or, these lines will have their logarithms made negative as well, which were positive. The negative is derived merely from their position,⁶ the logarithm applies only to the [absolute] magnitude, which remains the same even on the negative side. Otherwise, the logarithms of negative numbers are impossible [magnitudes] in algebra.

3. Scholium. Convenience demands that in these tables, the arcs under 45 degrees and their complements to 90 degrees stand together. This causes a peculiar arrangement; namely that the beginning and the end stand together at the fore, and the middle stands in the last pages.

4. Scholium. It is incorrect that, in general, the logarithms of the sine and tangent of the angle 0 is also given as 0, since this arc's sine and tangent are certainly not = 1.

5. Scholium. Only the singular sine of 30 degrees, and the singular tangent of 45 degrees are rational numbers. Now $90 \cdot 60 [= 3600]$ sines, tangents, and logarithms for both are in the usual tables; also for the common [?] first 10,000 numbers. Among these, the tables only indicate with complete accuracy the whole sine and its logarithm, the sine of 30 degrees and the logarithms of 1, 10, 100, 1000, 10,000: this means that there are 10 completely true theorems among 31,600. The remaining 31,590 are all approximations, and upon them rest the measurement of the heaven and the earth, and almost the entirety of applied mathematics. How true it is that we, in our surest and loftiest knowledge, merely approach the truth, without ever fully reaching it.

³weder eines Menschen Arbeit

⁴Ausgaben

⁵hundert Billionen

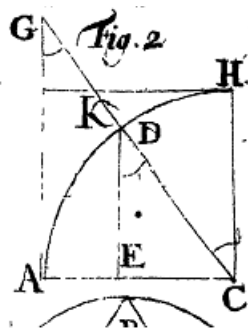
⁶i.e. which quadrant

On Right-Angled Triangles

3. Theorem.

Task.

In a right angled triangle CDE (figure 2), from the hypotenuse CD and one side DE , to find the acute angle:



Solution.

$CD : DE = r : \sin DCE$
 and $EDC = 90^\circ - DCE$
 or also $CD : DE = r : \cos CDE$.
 Thus $\sin DCE = \frac{r \cdot DE}{CD}$; $\cos CDE = \frac{r \cdot DE}{CD}$
 and if logarithms are used
 $\log \sin DCE = \log r + \log DE - \log CD =$
 $\log \cos CDE$.

Example. $CD = 5$, $DE = 4$ thus $\sin DCE =$
 $10,000,000 \cdot \frac{4}{5} = 8,000,000 = \sin 53^\circ 7'$ Thus $CDE = 36^\circ 53'$ -
 In logarithms $\log r + \log DE = 10,6020600$
 $\log CD = 0,6989700$
 $\log \sin DCE = 9,9030900 = \log \sin 53^\circ 7' +$

Corollary. EC can be found from geometry, theorem 15. Here, however, it can be found from the discovered angle in the following way:

$$r : \cos DCE = CD : CE \text{ or } CE = \frac{\cos DCE \cdot CD}{r}$$

and $\log \cos DCE + \log CD - \log r = \log CE$.

$$\text{In the example, } CE \text{ is } = \frac{6,001,876 \cdot 5}{10,000,000} = \frac{30,009,380}{10,000,000} = 3.0009380 :$$

However CE is actually = 3. That is to say, DCE is somewhat smaller than $36^\circ 53'$. Because one thus seldom finds the angle exactly as it stands in the table, one can make small errors, if one takes the angle in the table which comes nearest for the true one, and calculates further from there. Attentiveness to the circumstances must teach, whether these errors will be noticeable in the end results which one has in mind. However, it already becomes clear, that it is useful for accurate determinations to calculate lines from lines as often

In the triangle ECD , to find the side from the hypotenuse CD and an acute angle DCE .

Solution. $r : \sin DCE = CD : DE$, and $r : \cos DCE = CD : CE$.

6. Theorem. Problem.

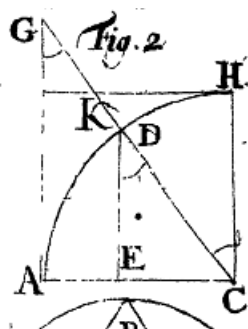
From a side EC and an acute angle ECD , to find the other side and the hypotenuse.

Solution. $r : \tan ECD = EC : ED$ and $\cos ECD : r = EC : CD$.

1. Scholium. If EC and EDC are given, one finds $ECD = 90^\circ - EDC$ and thus brings the given things under the circumstances here assumed.

2. Scholium. One could also use, instead of the first proportion, $\cos ECD : \sin ECD = CE : ED$. However, if it applies, a proportion should be used in which the radius appears, which shortens the calculation with numbers and with logarithms.

3. Scholium. Instead of the second proportion, $\tan ECD : \sec ECD = EC : CD$ can also be used. However, aside from the reason given in Scholium 2, secants are found in few tables, and their logarithms in fewer.



7. Theorem. Problem.

To find how many degrees are contained in an angle ACG which is drawn on paper.

Solution. One side is taken as a measure, which is divided into 1000 equal parts, $CG = 1000$, and drops the perpendicular GA onto the other side. This is measured using CG , and a sine is sought in the table (theorem 2, scholium 2), whose first three digits, if the last four are cut off, correspond to the discovered side of the perpendicular. The angle corresponding to this sine will be ACG .

In fact, in this way the angle is not found in individual minutes, but rather, for angles between half a degree to 50 degrees, it will be exact to between 4 and 5 minutes, and thus much more accurate than with the protractor, and at least equally as sharp for larger angles.

Corollary. To construct an angle, which is given in degrees, at a point C on a line AC ; one cuts the last four digits off the cosine and sine, and finds the first three digits of the cosine using CA as the measure [??] and one erects a perpendicular through A , and on it takes $AG =$ to the first three digits of the cosine.