Anfangsgründe
Volume I Section 1
Spherical Trigonometry

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1761

Translated from the German by Sky Shields

1. Definition.

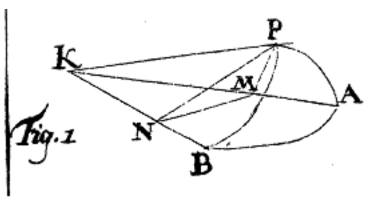


Fig. 1

A triangle that consists of three arcs BP, PA, AB, of great circles as in fig. 1, is called a **spherical triangle**. The arcs are called its **sides**, and the spherical angles (geometry, theorem 52., corollary 4.) its angles. **1. Corollary.** If one draws lines from the intersection points of the arcs to the center, K, of the sphere, then the sides of the triangle the measure of angles PKB, PKA, AKB. From the endpoint of one side (P, fig.4) the perpendiculars PM, PN, fall upon the plane of the others, AKB, and upon the intersection BK of both planes; thus is $PNM = B$ (geometry, theorem 45, corollary 6) Similarly, $PQM = A$.

Scholium. For the sake of brevity I will indicate by the arc BA the magnitude of this arc in degrees; but by **the plane** BA will I indicate the plane of the entire great circle of which BA is an arc and BKA is a section.

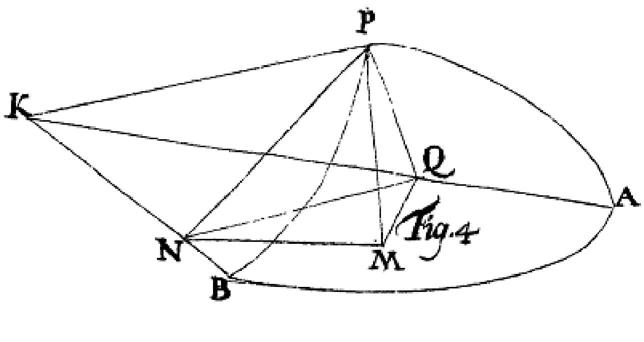


Fig. 4

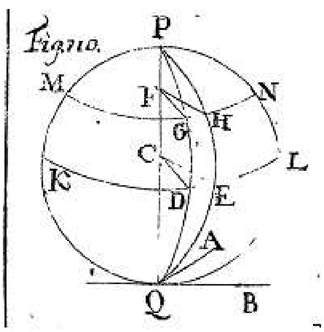
2. Corollary. The three planes BP, PA, BA, enclose a solid angle at the center of the sphere. (geom.

part II, def. 3) It is the point of a sort of pyramid, whose curved base would be the surface of a spherical triangle. Thus what geometry shows about the plane angles which enclose solid angle applies to the sides of a spherical triangle.

Scholium. This teaching I must omit here, because I have not sufficiently dealt with solid angles. In that teaching belongs[1], that in each spherical triangle the sum of two sides is greater than the third and the sum of all three is less than the whole circle. From Euclid Book. XI., theorems 20, 21.

3. Corollary. If the side PA were to lie in the plane through PM and K, fig. 1, then it would stand perpendicularly on the plane KB (geom. theorem 47.) and the triangle PAB is called **right angled**; PA its **perpendicular**; BA its **base**, or **the other perpendicular**; PB the **hypotenuse**.

4. Corollary. If N, M, K were to fall upon the same point, then could B and A both be right angles, since P is then the pole of circle BA (geom. theorem 52, corollary 2).



5. Corollary. Therefore a spherical triangle DPE (geom. fig. 110) can, depending on variations in its angles and pole, have three right angles or two right angles and an obtuse angle. Hence, what geom. theorem 13 teaches about plane triangles, does not hold here, wherefore this trigonometry will be much more extensive than that of the plane.

6. Corollary. If three great circles intersect one another, three of their intersection points can always be taken such that each side of the spherical triangle is smaller than 180 deg. (geom. theorem 49, corollary 5.) The following discussion shall be of these triangles only.

2. Definition.

The science of using three angles or sides of a spherical triangle to calculate those remaining, is called **spherical trigonometry**.

[1] *Dahin gehört*

On Right-Angled Triangles

1. Theorem.

In fig . 1, we have from Definition 1, corollary 3.:

I; $r : \sin BP = \sin B : \sin PA$ II; $r : \sin BA = \tan B : \tan AP$

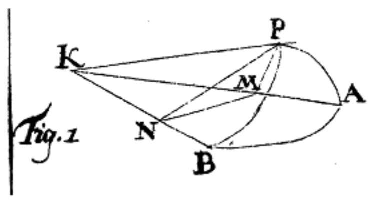


Fig. 1

Proof. The perpendicular PM falls here in the plane PKA (geometry, theorem 47, corollary 2) thus, for the radius of the sphere, is $PN = \sin PB$, $PM = \sin PA$; but $PN : PM = r : \sin B$ (def. 1 cor. 1.) therefore $r : \sin B = \sin BP : \sin PA$; for I.

Further $KM : MP = r : \tan PA$[2]
$MP : NM = \tan B : r$, thus (arith. V. 51.)[3]
$KM : NM = \tan B : \tan PA$
but $KM : NM = r : \sin BA$; thus is II also proven.

1. Corollary. In a right-angled triangle, the right angle is always given. If yet two more things are then given, a third can be immediately sought from them. These three things, which appear in the question, are called the terms of the question (*terminos quaestionis*;), and the following combinations can occupy those places; They can be α) three sides; β) 2 angles and an hypotenuse; γ) 2 angles and a perpendicular; δ) 2 perpendiculars and an angle; ϵ) 1 hypotenuse, a perpendicular, and an adjacent angle; ζ) 1 hypotenuse, a perpendicular and one opposite angle.

2. Corollary. From I, ζ; and from II, δ will be solved. It will be clear, however, that each of these proportions answers three questions; because each term of the questions can be the sought one; for this one simply uses the proportions in such a way, that that which is sought finally arises.

3. Corollary. However the remaining questions do not follow directly from I or II. One extends (fig. 2.) BP, BA until BN, BM become quadrants; thus are M,N, right angles; and the arcs AP, MN, intersect each other in the pole of BA, D; (geom. theorem 52, cor. 5) thus are PN, DP the complements of BP, PA, (geom. theorem 52, cor 2) and D, DN the complements of BA, B; (geom. theorem 52, cor. 4) Thus it follows from I that in the triangle DPN, $r : \sin DP = \sin D : \sin PN and r : \sin DP = \sin P : \sin DN$, and from II it follows that $r : \sin DN = \tan D : \tan PN$ and

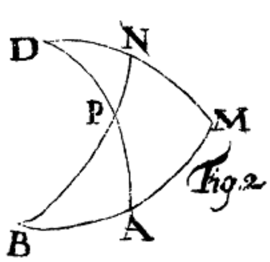


Fig. 2

[2] $\cos PA / \sin PA = 1 / \tan PA$
[3] $\sin B / \cos B = \tan B$

$r : \sin PN = \tan P : \tan DN$. In both of these proportions, one expresses the terms such that they refer to triangle BPA, thus giving

III; $r : \cos PA = \cos BA : \cos BP$ for α

IIII; $r : \cos PA = \sin P : \cos B$ . . . γ

V; $r : \cos B = \cot BA : \cot BP$ . . . ϵ

VI; $r : \cos BP = \tan P : \cot B$ . . . β

These six proportions thus satisfy all questions. Some of them are also expressed differently through trig. def. 5, cor. 2; namely

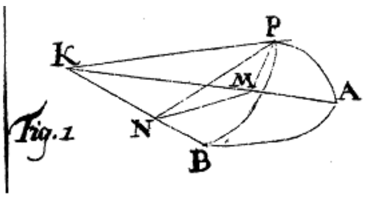
from II; we have VII; $r : \sin BA = \cot AP : \cot B$.

and from V; VIII; $r : \cos B = \tan BP : \tan BA$.

from VI; VIIII; $r : \cos BP = \tan B : \cot P$.

4. Corollary. In order to avoid the difficulty of assigning the appropriate proportion to every case of each question, the following table can be used. One need simply write, on each right triangle which is encountered, the same letters as on those with which we have just dealt. In this way one always finds in the table the proportion which solves any question which appears. In most cases, B and P can also be interchanged, in order that two separate rows can be made in the table. In all proportions it is the radius which facilitates the calculation. However the attempt has also been made, as often as possible, to make it the first term, and that is the reason for the arrangement in cases 9, 12, and 15. That the table however contains all possible cases, is demonstrated in the following manner. If each term of the question is made into that which is to be sought, α gives the two cases 14 and 1; since it is irrelevant which perpendicular is sought. Just as many are given by β, namely 15 and 3. From γ arise the three, 16, 7, 11. From δ equally as many, 2, 8, 9. Similarly 12, 4, 6, from ϵ, and 13, 5, 10, from ζ.

Given	Sought	Proportion
BA; PA	BP	$r : \cos BA = \cos PA : \cos PB$
BA; PA	B	$r : \sin BA = \cot PA : \cot B$
BP; P	B	$r : \cos BP = \tan P : \cot B$
BP; P	PA	$r : \tan BP = \cos P : \tan PA$
BP; B	PA	$r : \sin BP = \sin B : \sin PA$
BA; B	BP	$r : \cos B = \cot BA : \cot BP$
BA; B	P	$r : \cos BA = \sin B : \cos P$
BA; B	PA	$r : \sin BA = \tan B : \tan PA$
PA; B	BA	$r : \cot B = \tan AP : \sin AB$ (because $\cot AP = \frac{r^2}{\tan AP}$)
PA; B	BP	$\sin B : \sin PA = r : \sin BP$
PA; B	P	$\cos PA : r = \cos B : \sin P$
BA; BP	B	$r : \cot BP = \tan BA : \cos B$ (because $\tan BP = \frac{r^2}{\cot BP}$)
PA; BP	B	$\sin BP : r = \sin PA : \sin B$
PA; BP	BA	$\cos PA : r = \cos BP : \cos BA$
B ; P	BP	$r : \cot P = \cot B : \cos BP$ (because $\tan B = \frac{r^2}{cot B}$)
B ; P	PA	$\sin P : r = \cos B : \cos PA$



5. Corollary. If all sides of a triangle are not smaller than quadrants, the larger ones have their cosines and tangents negative, the sines however remain always positive, (Definition 1, corollary 6; and trig. Definition 2, corollary 7; Definition 3, corollary 2; Definition 4, corollary 1; Definition 5, corollary 3) from which one can usually see at once whether that which is to be sought is greater or smaller than a quadrant. E.g. in the first case the hypotenuse is smaller than a quadrant, if both sides are either smaller or larger than a quadrant, but larger if one is larger and the other smaller than a quadrant. These rules on account of the signs, combined with those of multiplication and division, thus comprise a quantity of theorems, which will be further proven in another way in Sphaerics, although it is not unhelpful to make known the way in which one can derive them from the positions of planes, and to realize how these deductions[4] can, at their foundations, be one and the same with the conclusions[5] derived from the signs. Regarding logarithms, see trigonometry, section 2, scholium

[4] *Schlüsse*
[5] *Folgerungen*

2. It is, however, important to derive these theorems from the signs here, since otherwise each solution contains an ambiguity because, signs set aside, the same tangents, cosines, etc., can belong to multiple angles or arcs, and here, to be sure, two different types. For the sake of brevity, I intend to call angles as well as arcs acute or obtuse, accordingly as they are under or over 90 degrees. And if an arc and an angle, or two arcs, or two angles, are both acute, or both obtuse, they shall be called **similar**[6], however when in such a pair one is obtuse and the other acute, they shall be called **dissimilar**[7]. Thus, from the proportions in the theorems and in corollary 3, flows the following:

6. Corollary. An oblique angle and its opposite perpendicular are always similar; from II, since $\sin BA$ is always positive, thus is $\tan AP$ positive or negative, depending on whether $\tan B$ is. That is, AP is acute or obtuse, depending on whether B is.

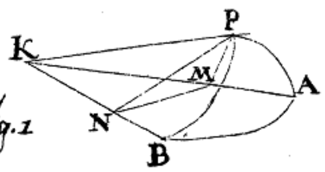


Fig. 1

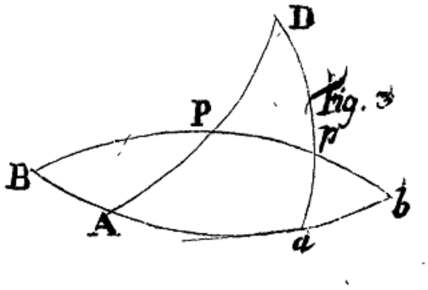
7. Corollary. If both angles are acute or obtuse, then so are both perpendiculars. If one angle is acute, and the other obtuse, so are their respective perpendiculars.

8. Corollary. If both perpendiculars are of the same type [acute or obtuse], the hypotenuse is then acute; and it is obtuse when the two perpendiculars are dissimilar. (III.) Conversely, the perpendiculars are similar or dissimilar, accordingly as the hypotenuse is acute or obtuse.

9. Corollary. The hypotenuse is acute or obtuse, according to whether an angle is similar or dissimilar to its adjacent side. (V.)

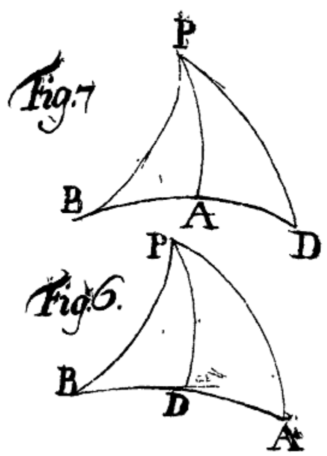
10. Corollary. One angle is acute or obtuse, depending upon whether the other is similar or dissimilar to the hypotenuse. (VI)

[6] *gleichartig*
[7] *ungleichartig*



11. Corollary. Similar observations indicate, as corollary 5 has shown in the first case, whether that which is sought is acute or obtuse, in all cases, with the exceptions of 5; 9; 10; 11; 13. The ambiguity in 5 and 13 is decided by corollary 6., however 9, 10, and 11 can not be determined purely from the given magnitudes. To wit, if in figure 3, from b, the great circle arcs BP, BA will be taken equal to the arcs bp, ba of the same circles; it arises that $pa = PA$, and thus from completely equal given things [BP,BA and bp, ba] can the things sought in these three cases [9, 10,11] also be Ba, Bp, and Bpa[8]. This case is here, what is section 12 in the plane trigonometry.

12. Corollary. If in a triangle (figures 6, 7), $BP = 90^o$, one thus also takes $BD = 90^o$, since then $DA = \pm BA \mp 90^o$, $DPA = \pm BPA \mp 90^o$, and $DP = B$ (geometry, section 52, corollary 2). Thus can the solution of a triangle, of which one side is a quadrant, always be reduced to the solution of a right angled triangle. And hence, one has the following rules for the acute triangles which are only relevant where no side is a quadrant; however they are also true when this condition is met.



[8]Which fulfill the proportions as well as BA, BP, and P, which are sought in 9, 10, and 11 respectively.

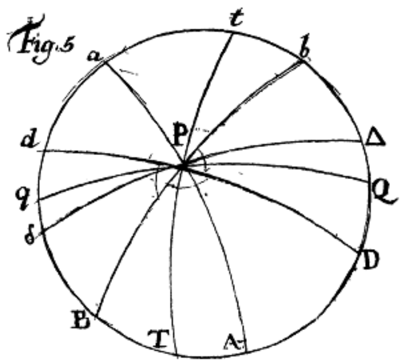
On Oblique Triangles

Section 2 [?]

Problem.

To determine the circumstances, under which the perpendicular shall fall within or without an oblique triangle.

Solution.



1. Upon the complete circle from figure 5, which I will call the **base**, the half circle APa stands perpendicular. Along this is taken an arbitrary point P, and a half circle DPd is drawn through it, obliquely [to APa]. This makes with the base, clearly [*ohnfehlbar*] two acute, and two obtuse angles (geometry, section 52, corollary 4). The former shall be ADP, AdP, and the latter, aDP, adP. Thus, PA is acute, Pa obtuse (section 1, corollary 6), or the acute arc of a perpendicular circle would fall between the two acute angles of the oblique [triangle? intersection?], the obtuse between the obtuse.

2. Another half circle BPb is laid obliquely; such that the angles ABP and ADP are acute, abP, adP obtuse, and if BA is taken acute, BP is also acute (section 1, corollary 8).

3. The figure thus presents us the four triangles: I) BPD, both angles acute at the base; II) bPd, both obtuse; III)bPD with D obtuse and b acute; and IIII) BPd, with B obtuse and d acute. The acute perpendicular falls within I, and the obtuse within II (I; 2) or if the angles on the base are similar, the perpendicular falls within the triangle, and is similar to them. Should they be dissimilar however, it falls outside the triangle, and is dissimilar to the angle which lies nearest it; PA to PDb; Pa with PdB.

4. If $r = 1$ (trigonometry, section 19, scholium) then (section 1, case 2) $\cot PDA = \cot PA \cdot \sin AD$. The first factor remains unchanged, and the second grows, if AD grows, and is positive. Therefore $\cot PDA$ also grows under these circumstances, and is positive. Therefore, diminish the acute angle PDA; this continues until D falls on Q, such that $AQ = 90^o$, or $\sin AQ = 1$; since $PQA = PA$ because both things are similar (section 1, corollary 6) D moves still further past Q to Δ, thus $\sin A\Delta$ wanes, while its obtuse arc grows. Thus $\cot P\Delta A$ also decreases, i.e., this angle closes further until Δ falls on b, and $\sin A\Delta = \sin AB$ since the angle Δ becomes $= b = B$.

5. Now, because d advances at the same time as D, the aforementioned obtuse angle in triangle bdP of the previous completion/complement [*Ergänzung*] grows to $2R$, while d moves from a to q, since it becomes $= Pa$, and decreases further from there.

6. In the triangle bDP, the obtuse angle grows while the side bD shrinks until it becomes bQ. From there it shrinks it shrinks until it differs infinitely little from Pba, if Δ comes infinitely near b.

7. It is the same with $\cos PD = \cos PA \cos DA$ (section 1, case 1) where, again, the first factor remains unchanged, but the second is initially positive and shrinks, while D moves away from A. That thus $\cos Pd$ is also positive, and shrinks, or the arc PD is acute and shrinks. In Q, $\cos AQ$ becomes $= 0$, thus $PD = 90^o$, subsequently is $\cos A\Delta$ negative, and grows along with $A\Delta$. Therefore, the obtuse arc $P\Delta$ grows until Δ falls on b.

8. In triangle II, (3) the arc Pd thus decreases constantly, while d moves from a, through q, to B. It is initially obtuse, until q, and afterwards acute.

9. In triangle III (3), PD changes as 7 indicates.

10. [*Wofern*] $AD < AB$, one thus takes $AT = AD$, and $PD = PT$ (section 1, case 1) and $PDA = PTA$ (section 1, case 2) thus $BTP = 2R - PDA$.

11. [*alsdenn*] is PT or $PD < BP$, however, as becomes clear, if one lets T move from A to B as before (7) and D from A to b.

12. In the triangle I (3) BP, B, PD can thus be given, and indeed, a triangle PBT like III. From those very same given things can be made if $PD < BP$, however not otherwise, (10; 11). And in the latter case, BDP is always acute (4).

13. From the given things bP, b, Pd, in triangle II, bPt can also become [?] if $Pd > Pb$ (8) however not otherwise, and in the latter case Pdb is obtuse.

14. From PbD, Pb, PD, the triangle PTb can also be made, if $PD < 2R - Pb$ (10, 11).

15. From PBd, PB, Pd, also, the triangle PtB [can be made] if $2R - Pd < PB$.

16. If W signifies generally an angle, l the side adjacent, L the side opposite, w the angle is opposite side l, and B, BP, PD, and D, the same as in triangle 1, then if

W is	and **l** is	and **L** is	then **w** is	
acute	acute	smaller than l	ambiguous	12
		larger than l	acute	
obtuse	obtuse	larger than l	ambiguous	13
		smaller than l	obtuse	
acute	obtuse	smaller than $2R - l$	ambiguous	14
		larger than $2R - l$	obtuse	
obtuse	acute	larger than $2R - l$	ambiguous	15
		smaller than $2R - l$	acute	

17. In 4 let $A\Delta = 2R - AD$; therefore $PDA = P\Delta A$ (4) and $-\cos DA = \cos \Delta A$, thus $PD + \Delta P = 2R$ (7). Now, the greatest is $A\Delta = 2R - AB$. Thus, the smallest AD, of which D belongs to a here determined Δ, is $= AB$. But for $AD = AB$, $ADP = ABP$ (10) thus for $AD > AB$, $ADP < ABP$ (4) because AB is acute (2) and there is thus a more acute AD [?]. However AD is as much more acute as $A\Delta$ is obtuse, since the latter ends [*sich endigen*] always within an arc which is $= AQ - AB$, and the former ends [*sich endigen*] within the equal arc $Ab - AQ$. To each of these AD belongs an $A\Delta$ since $A\Delta P = ADP$, and $AD + A\Delta = 2R$. However, to each of these AD belongs an $ADP < ABP$, and each $ADP > ABP$ belongs to an $AD < AB$ (4). If thus BP, PBD, and BDP are given, and the last angle is smaller than the first, there are two triangles which contain these given things. The second side is $P\Delta = 2R - PD$. However, if the last angle is larger than the first, there is only one such triangle, whose side is acute (7).

18. If bP, b, and d are given in triangle II (3), the third side can be Pd or $P\Delta$, [*wofern*] in triangle I the third side can be either PD or $P\Delta$ (from 17). I.e., [*wofern*] $2R - Pdb < 2R - Pbd or Pbd < Pdb$. However if this is not, there is only an obtuse Pd.

19. If bP, b, and D are given in triangle II, two sides appear [*stattfinden*] if $2R - PDb < PbD$, otherwise there is only the acute side PD.

20. If PB, B, and d are given, two sides appear [*stattfinden*], if PdB is $< 2R - PBd$, otherwise, only one obtuse Pd.

21. This gives, therefore, as in 16, the following table:

W is	and **l** is	and **w** is	then **L** is	
acute	acute	smaller than W	ambiguous	17
		larger than W	acute	
obtuse	obtuse	larger than W	ambiguous	18
		smaller than W	obtuse	
acute	obtuse	larger than $2R - W$	ambiguous	19
		smaller than $2R - W$	acute	
obtuse	acute	smaller than $2R - W$	ambiguous	20
		larger than $2R - W$	obtuse	

22. Because $AD = ad$, $A\Delta = a\delta$, thus this arc is ambiguous in the ambiguous cases; otherwise it is always acute.