Derivation of the Biquadratic Fundamental Theorem from the Theory of the Leminscatic Function, together with remarks on Multiplication and Transformation Formulas

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1.

If $a+bi=m$ signifies any one uneven complex whole number, and $a^2+b^2=p=N(m)$, then the integral of the differential equation

$$(1.)\ \frac{\delta y}{\sqrt{(1-y^4)}}=(a+bi)\cdot\frac{\delta x}{\sqrt{(1-x^4)}},$$

if y should dissapear conjointly with x, of the form:

$$(2.)\ y=x\frac{(A_0+A_1x^4+A_2x^8+....+A_{\frac{1}{4}(p-1)}x^p-1)}{(1+B_2x^4+B_2x^8+....+B_{\frac{1}{4}(p-1)}x^p-1)}=\frac{U}{V},$$

where the coefficients $A_0, A_1 etc. B_1 etc.$ are *whole complex* numbers. In this form of the integrals the numerator and denominator will not have algebraic and not even numerical common denominators. It follows next, to find the yet unknown whole coefficients, or at least to discover their characteristics, which becomes necessary here.

It is easily demonstrated, that $A_0=m$; if A_0 is the value of $\frac{y}{x}$ for $x=0$,thus $A_0=\frac{\delta y}{\delta x}$ for $x=0$; then y dissapears conjointly with x, and the differential equation (1.) for $x=0$, $\frac{\delta y}{\delta x}=m$ is yielded.

If in (1.) and (2.) x is set equal to $\frac{1}{\mu\xi}$, and y to $\frac{1}{\eta}$, and employing the whole number μ in the appropriate manner, thus is obtained

$$\frac{\delta\eta}{\sqrt{(1-\eta^4)}}=m\frac{\delta\xi}{\sqrt{(1-\xi^4)}},$$

$$\eta=i^\mu\eta\frac{B_{\frac{1}{4}(p-1)}+B_{\frac{1}{4}p-5}\xi^4+....+B_1\xi^{p-5}+\xi^p-1}{A_{\frac{1}{4}(p-1)}+A_{\frac{1}{4}p-5}\xi^4+....+A_1\xi^{p-5}+\xi^p-1}$$

consequently also

$$(3.)\ y = i^{\mu} x \frac{B_{\frac{1}{4}(p-1)} + ... + x^{p-1}}{A_{\frac{1}{4}(p-1)} + ... + A_0 x^{p-1}}$$

is an integral of a differential equation (1.), and since these likewise disappear conjointly with x, consequently they must be identical with (2.), in addition the numerator and denominator will not have common denominators, thus the numerator and denominator of the new and old y out of (3.) and out of (2.) coincide when individually compared to one another, notwithstanding the complex unit. Such a comparison yields:

$$(4.)\ A_0 = i^{\nu} B_{\frac{1}{4}}(p-1), A_1 = i^{\nu} B_{\frac{1}{4}}(p-?), ...A_{\frac{1}{4}(p-?)} = i^{\nu} B_1, A_{\frac{1}{4}(p-1)} = i^{\nu}$$

The exponent ν is a yet undetermined whole number. To determine it approximately and to give rise to the following inquiry with greater ease, be common $\varphi(t)$ that funciton of t, which dissapearls conjointly with t and satisfies the differential equation:

$$\frac{\delta\varphi(t)}{\delta t} = \sqrt{(}1 - \varphi(t)^4).$$

Furthermore is set:

$$\omega = 4^1 \int_0 \frac{\delta z}{\sqrt{(1-z^4)}} = \frac{\Gamma\frac{1}{4}^2}{\sqrt{(2\pi)}}, \text{ such that } \varphi\frac{\omega}{4} = 1.$$

The function $\varphi(t)$ is complex periodic, and $\varphi(t + k\omega) = \varphi(t)$, if k signifies any whole complex number. The fundamental characteristics of the function φ are given through both of the equations:

$$\varphi(it) = i\varphi(t) \text{ and}$$

$$\varphi(t+t') = \frac{\varphi(t)\sqrt{(1-\varphi(t')^4)}}{+}\varphi(t')\sqrt{(1-\varphi(t)^4)}1 + \varphi(t)^2\varphi(t')^2.$$

Herefrom and from the mentioned periodicity, which springs immediately from the fundamental equations, if one easily differentiates the formula:

$$\varphi[(2\alpha + 1 + 2\beta i)\frac{\omega}{4}] = (-1)^{\alpha+\beta},$$

if α and β are two real whole numbers. The integral (2.) of the differential equation (1.) can now be written in the following form:

$$\varphi(mt) = \varphi(t) \cdot \frac{A_0 + A_1\varphi(t)^4 + ... + A_{\frac{1}{4}(p-1)}\varphi(t)^{p-1}}{1 + B_{\varphi}(t)^4 + ... + B_{\frac{1}{4}(p-1)}\varphi(t)^{p-1}}.$$

To now determine the approximate value of ν in (4.), the special value $\frac{\omega}{4}$ is set in place of t, so that $\varphi(t) = 1$: since for this value, if still a is assumed uneven

$= 2\alpha + 1$, b even $= 2\beta$, after which the just now mentioned $\varphi(mt) = \varphi(m \cdot \frac{\omega}{4})$ is transformed into $(-1)^{\alpha+\beta}$, then from the just now written formula

$$(-1)^{\alpha+\beta} = \frac{A_0 + A_1 + ... + A\frac{1}{4}(p-1)}{1 + B_1 + ... + B_{\frac{1}{4}(p-1)}}$$

is obtained for a value of m whose uneven real part, the coefficient of i is uneven, thus appears, if the value of B from (4.) is inserted for A_0, A_1 etc:

$$(-1)^{\alpha+\beta} = \frac{B_{\frac{1}{4}(p-1)} + B_{\frac{1}{4}(p-5)} + ... + B_1 + 1}{1 + B_1 + ... + B_{\frac{1}{4}(p-5)} + B_{\frac{1}{4}(p-1)}}, which = i^{\nu},$$

because the numerator and denominator are here equal to one another and cannot disappear conjointly; consequently $i^{\nu} = (-1)^{\alpha} + \beta$.

For all cases sufficing for it, m *primary*, according to the sense(importance) of the words assigned to *Gauss*, i.e. taken $= 1(mod.2 + 2i)$, because all possible uneven complex numbers can be derived from the primary ones through muliplication with the complex units $\pm 1, \pm i$. For primary values of m $\alpha + \beta$ is even, thus $i^n u = +1$, consequently the multiplication formulas for a *primary* value of m, *but only for one such*, will be of the form:

$$(5.)\ \varphi(mt) = \varphi(t)\frac{m + A_1\varphi(t)^4 + ... + \varphi(t)^{p-1}}{1 + A\frac{1}{4}(p-5)\varphi(t)^4 + ... + m\varphi(t)^{p-1}},$$

$$y = \frac{mx + A_1x^5 + ... + x^p}{1 + A\frac{1}{4}(p-5)x^4 + ... + mx^{p-1}} = \frac{U}{V}.$$

We come now to a theorem, which appears to form the foundation for all applications of the lemniscate divisions to number theory and which is so remarkable, since it can be derived from the entirely simple differential equation. It is the following:

"If m is a *two-termed* complex *prime number*, then p is a real prime number $\equiv 1(mod.4)$, thus all coefficients of the numerators U except for the last are divisible by m, just as all the coefficients of the denominators V except for the first are divisible by m."

In point of fact, if $y = \frac{U}{V}$ is inserted in the differential equation (1.), thus appears,

$$V\frac{\delta U}{\delta x} - U\frac{\delta V}{\delta x} = m - \frac{\sqrt{(V^4 - U^4)}}{\sqrt{(1 - x^4)}}.$$

Hence, on the one hand it follows, that the term $\frac{\sqrt{(V^4 - U^4)}}{\sqrt{(1 - x^4)}}$ is identical to a whole function of x^4, which we will denote by T. On the other hand $V^4 - U^4$ is a whole function of x^4 with *whole* complex coefficients, whose constant element $= 1$, and which disappears for $x^4 = 1$, thus $V^4 - U^4$ is divisible by $1 - x^4$

and $\frac{(V^4-U^4)}{(1-x^4)} = T^2$ is a whole function of x^4 with *whole* coefficients, whose constant element(first coefficeint) = 1; consequently must also T itself, which already would be found as a whole function, have nothing but *whole* coefficients.

Since we have

$$V\frac{\delta U}{\delta x} - U\frac{\delta V}{\delta x} = mT$$

it follows therefrom

$$V\frac{\delta U}{\delta x} - U\frac{\delta V}{\delta x} \equiv 0(mod.m),$$

a congruence, which is understood in the sense that, each coefficient is divisible by m according to inverse order of the power of x. If it is effectively developed, according to the formulas

$$\frac{\delta U}{\delta x} = A_0 + 5A_1x^4 + 9A_2x^8 + 13A_3x^{12} + ...$$

$$\frac{\delta U}{\delta x} = 4B_1x^3 + 8B_2x^7 + 12B_3x^{11} + ...,$$

thus appears

$$V\frac{\delta U}{\delta x} - U\frac{\delta V}{\delta x} = A_0 + ({}_3A_0B1 + 5A_1)x^4 + (-7A_0B_2 + A_1B_1 + 9A_2)x^8$$
$$+ (-11A_0B_3 - 3A_1B_2 + 5A_2B_1 + 13A_3)x^{12} + etc.$$

Since here every single coefficient should be divisible by m, and since for a *two-termed prime number* m the numerical factors $5, 9, 13, ..., p-4$ do not go into m, then it is seen, if the already acquired congruences will become shared with each following, that $A_0, A_1, A_2, ...A\frac{1}{4}(p-5)$ must necessarily be divisible by m. For example, A_0 therefore as $-3A_0B_1 + 5A_1$ is divisible by m, then also $5A_1$; but 5 is not divisible by m, then must it be A_1; in addition m goes into $-7A_0B_2 + A_1B_1 + 9A_2$, but also in A_0 and A_1, as now found, consequently also in $9A_2$, since 9 is not divisible by m, and forward in the same manner. If the last coefficients can not be reduced further to take on the form $A\frac{1}{4}(p-1)$, since very well $pA\frac{1}{4}(p-1)$ must be and can be divisible by m, unless $A\frac{1}{4}(p-1)$ requires it. Similarly little can these conclusions employ, if m is a *single-termed* prime number: in these cases many of the numbers

$$5, 9, 13, ..., p-4$$

will be divislble by m.

We see also: "that for each primary *two-termed* complex *prime number* $m\varphi(mt)$ can be brought to the form:

$$(6.)\ \varphi(mt) = \varphi(t)\cdot\frac{m\mathbf{F} + \varphi(t)^{p-1}}{1 + m\mathbf{G}},$$

where $\mathbf{F}$ and $\mathbf{G}$ are *whole*functiones from $\varphi(t)^4$ with *whole* complex coefficients."

Example.

$$\text{for } m = -1+2i, p = 5, y\text{will} = \frac{(-1+2i)x + x^5}{1 + (-1+2i)x^4};$$

$$\text{for } m = 3+2i, p = 13, y \text{ will} = \frac{(3+2i)x + (7-4i)x^5 + (-11+10i)x^9 + x^{13}}{1 + (-11+10i)x^4 + (7-4i)x^8 + (3+2i)x^{12}},$$

and one has

$$7 - 4i = (3+2i)(1-2i), -11 + 10i = (3+2i)(-1+4i),$$

such that

$$\mathbf{F} = 1 + (1-2i)x^4 + (-1+4i)x^8,$$
$$\mathbf{G} = (-1+4i)x^4 + (1-2i)x^8 + x^{12}.$$

For $m = 1+4i, p = 17$ the coefficients of the numerators U are according to the progression:

$$1+4i, -20-12i = (1+4i)(-4+4i), -10+28i = (1+4i)(6+4i),$$
$$12 - 20 = (1+4i)(-4-4i), 1$$

Also in order to give an example therefrom, that the theorem for loses its validity, if the coefficients of the numerators are considered for $m = 5, p = 25$, which are

$$5, -62, -105, 300, -125, 50, 1$$

where -62 is not divisible by 5.

2.

After the exclusion of the terms divisible by the modulus, a system of complex residues for the (uneven, primary) modulus m can be classified into types of numbers, in the following way:

$$(7.)\ \left\{\begin{array}{l|l|l|l}
r_1 & ir_1 & -r_1 & -ir_1 \\
r_2 & ir_2 & -r_2 & -ir_2 \\
r_3 & ir_3 & -r_3 & -ir_3 \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
r_{\frac{1}{4}(p-1)} & ir\frac{1}{4}(p-1) & -r\frac{1}{4}(p-1) & -ir\frac{1}{4}(p-1)
\end{array}\right.$$

If any one term of the system of residues is taken for r_1 [namely], then the four numbers of the first horizontal series of the schemas are incongruent relative to one another; if the system of the residues is not yet exhausted, then any one of the yet remaining[uebrig bleibenden] terms are taken for r_2, whereupon then the second horizontal series will be incongruent relative to one another and to those of the first; and then continuing along, until all the terms of the system of residues are exhausted.footnote: Incidentally it arises from this, that the norm of a complex number of the fourth roots of the unit is $\equiv 1 (mod..4)$; within the special cases for the 4th roots of the unit, this remark is indeed of little relevance, but it is interesting, that an analogous conclusion can be adopted with complex numbers from any power roots of unity.[needs correction] We call this classification: *the division of a system of residues into four associated parts*, and in short each of these four parts a *quarter-system of residues*. After any one type of this division has been dealt with, all remaing types can be derived from it, as the terms of each particular horizontal series is permuted cyclically, so that the number of types will be $= 4^{\frac{1}{4}(p-1)}$. The notion of a system of residues divided in this way, stated first by *Gauss*, is equally important as the concept of congruence itself, and based on the treatment of the kinds of numbers which behave towards associates similarly, such as congruent number to the same[gleichen]. Having said these few words, a *quarter system of residues* is every series of $\dfrac{1}{4}(p-1)$ numbers, from which, by addition of multiples of the modulus and through multiplcation with complex units, *all* numbers not divisible by the modulus can be generated, and each only once. After this preliminary remark we turn ourselves to the treatment of the roots of the equation $\dfrac{U}{x} = 0$. $\dfrac{U}{x}$ is a whole function of x^4 of the $\frac{1}{4}(p-1)$ten degrees, thus is had the equation $\frac{U}{x} = 0$, if in its $x^4 = z$ considered as the unknowns, i.e the equation

$$(8.)\ z^{\frac{1}{4}(p-1)} + A^{\frac{1}{4}(p-1)-1}_{\frac{1}{4}(p-5)} + ... + A_1 z + m = 0,$$

$\frac{1}{4}(p-1)$ roots z, which, since the equation (8.) following $\varphi(mt) = 0$ entails, evidently is given by the formula $z = \varphi(\dfrac{rw}{m}^4)$, in which by r is denoted the aggregate of the numbers $r_1, r_2, ...r_{\frac{1}{4}(p-1)}$ from the first vertical column of the schemas (7.). Since the coefficients of the equation (8.) is whole, thus also is each symmetrical whole function its roots thus "*each symmetrical function*" of $\frac{1}{4}(p-1)$ magnitudes

$$\varphi(\frac{r_1 w}{m}^4), \varphi(\frac{r_2 w}{m}^4), \varphi(\frac{r_3 w}{m}^4), \text{ etc.,}$$

whose aggregate will be denoted by $\varphi(\dfrac{rw}{m}^4)$, a *whole*complex number," and since the product of all roots of the positve or negative last coefficients are equal, depending on whether the degrees of the equation is an even or uneven number, thus is had namely

$$(9.)\ \prod \varphi(\frac{rw}{m})^4 = (-1)^{\frac{1}{4}(p-1)} \cdot m, \text{ or}$$

$$(9'.) \sqrt[4]{(-1)^{\frac{1}{4}(p-1)} \cdot m} = \prod \varphi(\frac{rw}{m}),$$

where the multiplications symbol $\prod$ refers to all values of r. The importance of the formula (9′.) consists therein, that it is the only[way] so far, by which the *fourth roots from a complex number* can be expressed, that is to say *rational*; it represents to the part of the place of the *Gaussian* formulas in the complex theory. –That here the saying incidentally means, m may be single-termed, or two-termed, if only primary.

3.

In order to proceed to the biquadratic residues, it is, with maintaince of the former nomenclature, m is any primary prime number, equivalent whether single-termed or two-termed, n in contrast is a primary and two-termed prime number, which is distinct from m; p and q are respectively the normes of m and n.–If all the $r's$ are multiplied with n, then the residues of the various products nr are situated in part among the r, and in part among the ir, $-r$, or $-ir$; thus it can generally be set;

$$nr \equiv i^{\mu} r' (mod.m),$$

where the various r', which correspond to the various r, are all located among the r, and are all different from one another [1], then coincides, if as well in another order, with all r. Because of the complex periodicity of the function φ and owing to $\varphi(it) = i\varphi(t)$:, from the just written congruence follows:

$$\varphi(\frac{nrw}{m}) = \varphi(\frac{i^{\mu} r' w}{m}) = i^{\mu} \varphi(\frac{r'w}{m}), i^{\mu} = \frac{\varphi(\frac{nrw}{m})}{\varphi(\frac{r'w}{m})},$$

consequently it is obtained, if the value of i^{μ} will be inserted in the congruence, in all cases:

$$nr \equiv r \cdot \frac{\varphi(\frac{nrw}{m})}{\varphi(\frac{r'w}{m})} (mod.m),$$

and since all r' coincide, if as well in another order, with all r, then through multiplication by all r appears:

$$n^{\frac{1}{4}(p-1)} \prod(r) \equiv \prod(r) \frac{\prod \varphi(nt)}{\prod \varphi(t)} (mod.m),$$

where for the facility of the letters t is written to represent $\frac{rw}{m}$. Now since none of the numbers r are divisible by m, thus it can be divided on either side with $\prod(r)$ and obtain

$$(10.)\ n^{\frac{1}{4}(p-1)} \equiv \frac{\prod \varphi(nt)}{\prod \varphi(t)} (mod.m),$$

what also can become written:

$$(10'.)\ \left[\frac{n}{m}\right] = \prod \frac{\varphi(nt)}{\varphi(t)},$$

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[1] Because nr_{σ} can never be $\equiv \pm nr_{\tau}$ or $\equiv \pm inr_{\tau}$, but r_{σ} would be $\equiv \pm r_{\tau}$ or $\equiv \pm ir_{\tau}$, which the latter however evidently contradicts the method of construction of our schemas (7.)

as soon as one denotes by $\left[\frac{n}{m}\right]$ that *complex unit*, to which the power $n^{\frac{1}{4}(p-1)}(mod.m)$ is congruent. To the expanded treatment of the analytical formula for $\left[\frac{n}{m}\right]$ in (10.) serves the result founded in §.1., according to which

$$\frac{\varphi(nt)}{\varphi(t)} = \frac{nF[\varphi(t)^4] + \varphi(t)^{q-1}}{1 + nG[\varphi(t)^4]},$$

because n is to be supposed as a two-termed primary prime number; F and G are whole functions of $\varphi(t)^4$ with whole complex coefficients. After this comes by means of (10′.)

$$(11.)\quad \left[\frac{n}{m}\right] \cdot \prod \left\{1 + nG\left[\varphi(\frac{rw}{m})\right]\right\} = \prod \left\{nF\left[\varphi(\frac{rw}{m})\right] + \varphi(\frac{rw}{m})^{q-1}\right\}.$$

Both of the products to the left and to the right in this formula, in which the multiplication from all values denoted by r, can, if the multiplication is carried out, pure algebraic powers orded according to n, and it apears, that then each coefficient of this expansion will equal a symmetrical function of the magnitudes $\varphi(\frac{rw}{w})^4$, then according to §2. will be equal to a whole complex number: so that one obtains

$$\left[\frac{n}{m}\right] \cdot \left\{1 + a_1 n + a_2 n^2 + ... + a_{\frac{1}{4}(p-1)} n^{\frac{1}{4}(p-1)}\right\} =$$
$$\prod \varphi(\frac{rw}{m})^{q-1} + b_1 n + b_2 n^2 + ... + b_{\frac{1}{4}(p-1)} n^{\frac{1}{4}(p-1)},$$

where a_1, a_2 etc.,b_1, b_2 etc. are complete whole complex numbers, namely complete symmetrical whole functions of the roots of the equation (8.). If here for $\prod \varphi(\frac{rw}{m})^{q-1}$ is set the value $\left[(-1)^{\frac{1}{4}(p-1)} m\right]^{\frac{1}{4}(p-1)}$ from (9.) and omitting from both sides the term divisible by n, thus one stands at the objective, then is obtained:

$$\left[\frac{n}{m}\right] \equiv (-1)^{\frac{1}{4}(p-1)\cdot\frac{1}{4}(q-1)} \cdot m^{\frac{1}{4}(q-1)} (mod.n), i.e.$$

$$(12.)\quad \left[\frac{n}{m}\right] = (-1)^{\frac{1}{4}(p-1)\cdot\frac{1}{4}(q-1)} \cdot \left[\frac{m}{n}\right].$$

This is the *Gaussian* reciprocity theorem for all cases excluding the one, if both numbers are real(single-termed). In these latter cases the theorem immediately follows without the furthering of from the *Fermat* theorem; then are m and n both real prime numbers $\equiv 3(mod.4)$, then m^2 and n^2 are their norms and one has

$$(\pm m)^{\frac{1}{4}(n^2-1)} = (\pm m^{\frac{1}{4}(n+1)})^{n-1} \equiv 1(mod.n), \text{ and}$$
$$(\pm n)^{\frac{1}{4}(m^2-1)} = (\pm n^{\frac{1}{4}(n+1)})^{m-1} \equiv 1(mod.m), \text{ then } \left[\frac{m}{n}\right] = \left[\frac{n}{m}\right].$$

In order to give (12.) the form of the reciprocity theorem, in which *Gauss* has demonstrated it, one may only note, that for $m = a + bi, p = a^2 + b^2, \frac{1}{4}(p-1) =$

$\frac{1}{4}(a^2-1)+\frac{1}{4}b^2=\frac{1}{4}(a^2-1)+(\frac{1}{2}b)^2$, and that a^2 as a square of an uneven number $\equiv 1(mod.8)$, consequently $\frac{1}{4}(a^2-1)$ is even, then $\frac{1}{4}(p-1)\equiv(\frac{1}{2}b)^2\equiv\frac{1}{2}b(mod.2)$. Since $a+bi$ is primary, then one has $\frac{1}{2}\equiv\frac{1}{2}(a-1)(mod.2)$, thus also $\frac{1}{4}(p-1)\equiv\frac{1}{2}(a-1)(mod.2)$; likewise if n is set $=c+di$, then in the same way $\frac{1}{4}(q-1)\equiv\frac{1}{2}(c-1)(mod.2)$, thus in (12.) in place of $\frac{1}{4}(p-1)\cdot\frac{1}{4}(q-1)$ can be written $\frac{1}{2}(a-1)\cdot\frac{1}{2}(c-1)$; then is obtained

$$\left[\frac{n}{m}\right]=(-1)^{\frac{1}{2}(a-1)\cdot\frac{1}{2}(c-1)}\cdot\left[\frac{m}{n}\right];$$

and this is the *Gaussian* form of the theorem.

It is yet to be remarked, that from the relationship of the formulas (9'.) and (10'.)

$$\begin{aligned}&\prod\varphi(\frac{rw}{m})=\sqrt[4]{((-1)^{\frac{1}{4}(p-1)}m)},.\\&\prod\varphi(\frac{nrw}{m})=\left[\frac{n}{m}\right]\cdot\prod\varphi(\frac{rw}{m},\text{ the following arises}\\&\prod\varphi(\frac{nrw}{m})=\left[\frac{n}{m}\right]\sqrt[4]{((-1)^{\frac{1}{4}(p-1)}m)}.\end{aligned}$$

The first of these three at last written formulas hold for a any primary m, both of the remaining in contrast hold only for prime numbers m; n is whole arbitrary, only not divisible by m; r pass through by the multiplication of a quarter system of residues $(mod.m)$.

4.

It may not be out of place here, to indicate a single method, according to which the coefficients in the numerator and denominator of the multiplications formula for the lemniscate can be denoted.—The complex whole number m will only be subject here to a single restriction, to be primary, i.e. $\equiv 1(mod.2+2i)$. Let

$$(1.)\ \frac{\partial x}{\partial t}=\sqrt{(1-x^4)}\text{ and}\frac{\partial y}{\partial t}=m\sqrt{(1-y^4)},$$

which both equations represent the figures of the uique [case] $\frac{\partial y}{\partial t}=\frac{m\sqrt{(1-y^4)}}{\sqrt{(1-x^4)}}$, thus is

$$y=\frac{U}{V}=\frac{x^p+B_1x^{p-4}+....+A_1x^5+mx}{mx^{p-1}+A_1x^{p-5}+....+B_1x^4+1}.$$

If one considers in the equation

$$U-Vy=0,i.e.(2.)\ x^p-my\cdot x^p-1+B_1x^p-4-A_1y\cdot x^p-5+....+mx-y=0$$

x as a function of y, then gives the root p of the equation, which we denote by

$$(3.)\ x_1,x_2,x_3,....x_p$$

p function of y, which completely satisfy the differential equation (1.). Each symmetrical function of the magnitudes x_1, x_2 etc. can be demonstrated by the coefficients of the equation (2.), and could inversely determine each symmetrical function of this magnitude in every direct way, thus also the coefficients of the equation (2.) would be had, and also the coefficients of the numerators and denominators A_1 etc. B_1 etc.

Since in the equation (2.) the coefficient of x^{p-1} is equal to $-my$, that of $xp-2$ is *zero*, then is the sum of the magnitude (3.) $=my$, the sum its combination by twos $=0$, consequently also the sum of its square $=m^2y^2$; this well known result can be written thus

$$(4.)\ \sum x = my, \sum x^2 = m^2y^2,$$

as generally the sum $\sum F(x)$ will be denoted by

$$F(x_1) + F(x_2) + ... + F(x_p).$$

By the continued differentiation of (4.)it will work, in determining all symmetrical combinations of the magnitude (3.); since in the meantime all symmetrical combinations in the *power sum* can be demonstrated, then it satisfies, to find the latter; it would be the power sum of the magnitude (3.) denoted by S_0, S_1, S_2 *etc*, so that

$$S_\mu = \sum x^\mu = (x_1)^\mu + (x_2)^\mu .... + (x_p)^\mu;$$
$$S_0 = 1 + 1 + ... + 1 = p.$$

One of the most essential principles in trigonometry, by which the *linear expressions* are attained, is in the expansion of powers of circular functions according to sines or cosines of multiple arcs; since these aids are lacking with elliptical functions, as *Jacobi* stated in his *fundamentals*; which are predicated on the potentiality, which higher differential quotients of elliptical functions by the power of this these functions, and inversely the powers are expressed by *linear* diferential quotients. As many a beautiful mathematical ideas remain because of the deficiency in unfruitful analytical labours, thus it also seems these principles, out of the applications made by *Jacobi*, have not been more utilized. Our problems here can become used by the just mentioned aid with success. Namely, we differentiate the equation (4.)μ times with respect to t, so that we obtain

$$(5.)\ \sum \frac{\partial^\mu x}{\partial t^\mu} = m\frac{\partial^\mu y}{\partial t^\mu}, \sum \frac{\partial^\mu (x^2)}{\partial t^\mu} = m^2 \frac{\partial^\mu (y^2)}{\partial t^\mu},$$

transformed for even values of μ the differential quotients to the right of the equal sighn in powers of y and then set the sums to the left of the formula, which the powers of x in the differential quotients of x or x^2, then also the power sums $\sum x^\mu$ expressed in the sums of differential quotients, thus we obtain directly the entire power sum S_μ as a function of y. If y is set $=0$, then the power sums of each magnitude are obtained herefrom, which the numerator of the multiplication formulas vanish. In fact, once the equation $\frac{\partial x}{\partial t} = \sqrt{(1-x^4)}$, 3 times, 5 times etc. with respect to t, thus comes

$$\begin{aligned}&\frac{\partial^2 x}{\partial t^2} = -2x^3, \frac{\partial^4 x}{\partial t^4 = -12x + 24x^5,}\\ &\frac{\partial^6 x}{\partial t^6} = 72 \cdot (7x^3 - 10x^7), etc.\end{aligned}$$

Further is obtained by the differentiation of the equation $\frac{\partial x}{\partial t} = \sqrt{(1-x^4)}$:

$$\begin{aligned}&\frac{\partial^2 (x^2)}{\partial t^2} = 2 - 6x^4, \frac{\partial^4 (x^2)}{\partial t^4} = 24(-3x^2 + 5x^6),\\ &\frac{\partial^6 (x^2)}{\partial t^6} = 144(-1 + 28x^4 - 35x^8), etc.\end{aligned}$$

One can derive each following equation from the forgoing with the aid of the formula

$$\frac{\partial^2 \psi}{\partial t^2} = (1-x^4) \cdot \frac{\partial^2 \psi}{\partial x^2} - 2x^3 \cdot \frac{\partial \psi}{\partial x}.$$

The even differential quotients of x, x^2 with respect to t thus become expressed by uneven and even whole functions of x, respectively; which this produces, letting the brevity of half be denoted by $(E.)$.

The same treatment of the equation $\frac{\partial y}{\partial t} = m\sqrt{(1-y^4)}$

$$(F.)\quad \left\{\begin{array}{ccr} \frac{\partial^2 y}{\partial t^2} & = & -2m^2y^3, \frac{\partial^4 y}{\partial t^4} = 12m^4(-y + 2y^5),\\ \frac{\partial^6 y}{\partial t^6} & = & 72m^6(7y^3 - 10y^7), etc,\\ \frac{\partial^2 (y^2)}{\partial t^2} & = & 2m^2(1-3y^4), \frac{\partial (y^2)}{\partial t^4} = 24m^4(-3y^2 + 5y^6),\\ \frac{\partial^6 (y^2)}{\partial t^6} & = & 144m^6(-1 + 28y^4 - 35y^8), etc. \end{array}\right.$$

These equations, which we denote by $(F.)$, given the even differential quotients of y and y^2 in respect to t are expressed by uneven and even whole functions of y, respectively, and become derived from eachother with the aid of the formula

$$\frac{\partial^2 \psi}{\partial t^2} = m^2(1-y^4)\frac{\partial^2 \psi}{\partial y^2} - 2m^2y^3\frac{\partial \psi}{\partial y}.$$

The equation $(E.)$ can resolve into $x^3, x^5, etc., x^4, x^6, etc.$, and then given the uneven powers of x expressed in the differential quotients of x, the even powers of x expressed in the differential quotients of x^2, namely:

$$(G.)\quad \left\{\begin{array}{cccccc} x^3 & = & -\frac{1}{2}\frac{\partial^2 x}{\partial t^2}, & x^5 & = & \frac{1}{2}x + \frac{1}{24}\frac{\partial^4 x}{\partial t^4}, etc.\\ x^4 & = & \frac{1}{3} - \frac{1}{6}\frac{\partial^2 (x^2)}{\partial t^2}, & x^6 & = & \frac{3}{5}x + \frac{1}{120}\frac{\partial^4 (x^2)}{\partial t^4}, etc. \end{array}\right.$$

Most conveniently the equation $(G.)$ can be established from each of both of the forgoing according to the formula

$$x^{\mu} = \frac{\mu-3}{\mu-1}x^{\mu-?} - \frac{1}{(\mu-1)(\mu-2)} \cdot \frac{\partial^2(x^{\mu-2})}{\partial t^2}.$$

This formula yields the equation $(G.)$ with greater ease in any amount, because according as it is only able, to get to x^{μ}, in which the expression for $x^{\mu-2}$ advances all diferentiations-exponenents of 2 units and that $\frac{1}{(\mu-1)(\mu-2)}$ multiples of this then obtains the expression subtracted from the $\frac{\mu-3}{\mu-1}$ multiples of the expression for $x^{\mu-4}$.

If in $(G.)$ the values $x_1, x_2....x_p$ for x and added, then the equation $(H.)$ is obtained

$$(H.)\quad \left\{ \begin{array}{lcllcl} \sum x^3 & = & -\frac{1}{2}\sum\frac{\partial^2 x}{\partial t^2}, & \sum x^5 & = & \frac{1}{2}\sum x + \frac{1}{14}\sum\frac{\partial^4 x}{\partial t^4}, etc, \\ \sum x^4 & = & \frac{1}{3}p - \frac{1}{6}\sum\frac{\partial^2(x^2)}{\partial t^2}, & \sum x^6 & = & \frac{3}{5}\sum x^2 + \frac{1}{120}\sum\frac{\partial^4(x^2)}{\partial t^4}, etc, \end{array} \right.$$

which according to the formula

$$\sum x^{\mu} = \frac{\mu-3}{\mu-1}\sum x^{\mu-4} - \frac{\partial^2 \sum x^{\mu-2}}{(\mu-1)(\mu-2)\partial t^2}$$

emerge from one another.— If now finallly in the last $(H.)$ is set according to (5.) for

$$\begin{array}{llll} \sum x, & \sum\frac{\partial^2 x}{\partial t^2}, & \sum \partial^4 x \partial t^4, & etc. \text{ respectively the values} \\ m\cdot y, & m\cdot\frac{\partial^2 y}{\partial t^2}, & m\cdot\frac{\partial^4 y}{\partial t^4}, & etc., \text{ and for} \\ \sum x^2, & \sum\frac{\partial^2(x^2)}{\partial t^2}, & \sum \partial^4(x^2)\partial t^4, & etc., \text{ respectively the values} \\ m^2 y, & m^2\frac{\partial^2(y^2)}{\partial t^2}, & m^2\cdot\frac{\partial^4(y^2)}{\partial t^4}, & etc. \text{ thus appears :} \end{array}$$

$$(I.)\quad \left\{ \begin{array}{lcllcl} \sum x^3 & = & S_3 = -\frac{1}{2}m\frac{\partial^2 y}{\partial t^2}, & S_5 & = & \frac{1}{2}my + \frac{m}{24}\cdot\frac{\partial^4 y}{\partial t^4}, etc., \\ S_4 & = & \frac{1}{3}p - \frac{m^2}{6}\cdot\frac{\partial^2(y^2)}{\partial t^2}, & S_6 & = & \frac{3m^2}{5}y^2 + \frac{m^2}{120}\cdot\frac{\partial^4(y^2)}{\partial t^2}, etc.,^3 \end{array} \right.$$

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and if the differential quotients of y converted to even powers of y, then one obtains $S_3, S_5, S_7, etc.$ expressed as uneven, S_4, S_6, S_8, etc expressed as even whole functions of y, e.g.

$$S_3 = m^3y^3, S_4 = \frac{1}{3}(p - m^4) + m^4y^4, etc.$$

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[3] The right side of the equation $(I.)$ is formed exactly the same, as the equation $(H.)$ and comes directly out of the latter, if the sum sign $\sum$ is ommitted to the right and writes throughout my in place of x, m^2y^2 in place of x^2.

After we have thus discovered the nature of the power sum S^{μ}, namely, that it is a whole function of y, which rises upto the μth degree, and merely contains uneven or even powers of y, depending on whether μ is uneven or even, then we will arrange a recursion formula, according as the each power sum can be calculated immediatly from the forgoing.—Since S_{μ} is a whole function of y of the μth degree, thus one obtains by the two-fold differentiation with respect to t

$$\frac{\partial^2 S_{\mu}}{\partial t^2} = \frac{\partial^2 S_{\mu}}{\partial y^2}(\frac{\partial y}{\partial t})^2 + \frac{\partial^2 S_{\mu}}{\partial t^2} \cdot \frac{\partial^2 y}{\partial t^2} = m^2(1-y^4) \cdot \frac{\partial^2 S_{\mu}}{\partial t^2} - 2m^2y^3 \cdot \frac{\partial^2 S_{\mu}}{\partial t^2},$$

where $\frac{\partial S_{\mu}}{\partial y}$ and $\frac{\partial^2 S_{\mu}}{\partial y^2}$ are likewise whole functions of y to the $\mu - 1th$ and $\mu - 2th$ degree, respectively. From the other side is also $\frac{\partial^2 S_{\mu}}{\partial t^2} = \sum \frac{\partial^2 (x^{\mu})}{\partial t^2}$; however

$$\begin{aligned}
\frac{\partial^2 (x^{\mu})}{\partial t^2} &= \mu(\mu-1)x^{\mu-2}(\frac{\partial x}{\partial t}^2 + \mu x^{\mu-1} \cdot \frac{\partial^2 x}{\partial t^2} \\
&= \mu(\mu-1)x^{\mu-2}(1-x^4) - 2\mu x^{\mu+2} \\
&= \mu(\mu-1)x^{\mu-2}, thus \\
\frac{\partial^2 S_{\mu}}{\partial t^2} &= \mu(\mu-1)S_{\mu} - 2 - \mu(\mu+1)S_{\mu+2};
\end{aligned}$$

this value of $\frac{\partial^2 (S_{\mu})}{\partial t^2}$ compared with that obtained a little while go, yields

$$m^2(1-y^4) \cdot \frac{\partial^2 (x^{\mu})}{\partial y^2} - 2m^2y^3 \cdot \frac{\partial S_{\mu}}{\partial y} = \mu(\mu-1)S_{\mu} - 2 - \mu(\mu+1)S_{\mu+2};$$

follows herefrom, if yet $\mu - 2$ is set to stand for μ:

$$(6.) \quad = \frac{\mu-3}{\mu-1}S_{\mu-4} + \frac{m^2}{(\mu-1)(\mu-2)}\left\{2y^3\frac{\partial S_{\mu-2}}{\partial y} - (1-y^4)\frac{\partial^2 S_{\mu-2}}{\partial y^2}\right\}.$$

this is the sought for recursion formula, according to which S_{μ} can be derived directly from $\S_{\mu-4}$ and $S_{\mu-2}$. Since it has already been found that $S_0 = p, S_1 my, S_2 = m^2y^2, S_3 = m^3y^3$, thus after this, the following power sum is easily derived:

$$\begin{aligned}
S_4 &= \frac{1}{3}S_0 + \frac{m^2}{2\cdot 3}(2y^3\frac{\partial S_2}{\partial y} - (1-y^4)\frac{\partial^2 S_2}{\partial y^2} \\
&= \frac{1}{3}p + \frac{m^2}{2\cdot 3}(4m^2y^4 - 2m^2(1-y^4)) = \frac{1}{3}(p-m^4) + m^4y^4, \\
S_5 &= \frac{1}{2}(m-m^5)y + m^5y^5, \\
S_6 &= \frac{3m^2-3m^6}{5}y^2 + m^6y^6, etc.e.g.yet \\
S_8 &= \frac{25p - 28m^4 + 3m^8}{3\cdot 5\cdot 7} + \frac{4}{5}(m^4-m^8)y^4 + m^8y^8, \\
S_{10} &= \frac{7m^2 - 9m^6 + 2m^{10}}{15}y^2\frac{5(m^6-m^{10})}{5}y^6 + m^{10}y^{10},
\end{aligned}$$

If the construction of this expression is considered, then one will easily the understand the truth of the following observation. The coefficients of the various powers of y in S_μ are whole functions of m, so that S_μ can be considered a whole function of both magnitudes m and y, and indeed it is found that only those powers both from m and from y, whose exponents $\equiv \mu(mod.4)$; the norm p only appears, if μ is divisible by 4, and indeed it is found to be only linear and neither multiplied by m or y; the coefficient of p in $S_{\mu 4}$ is

$$= \frac{1\cdot 5\cdot 9\cdot 13\cdots(4\mu-3)}{3\cdot 7\cdot 11\cdot 15\cdots(4\mu-1)},$$

and the [coefficient] of m^2y^2 in $S_{4\mu-2}$ is

$$= \frac{3\cdot 7\cdot 11\cdots(4\mu-5)}{5\cdot 9\cdot 13\cdots(4\mu-3)};$$

the coefficient of y^μ in S_μ is $= m^\mu$, that of $y^{\mu-4}$, $\frac{\mu m^{\mu-4}(1-m^4)}{10}$, that of $y^{\mu-8}$ likewise in S_μ,

$$= \frac{\mu m^{\mu-8}(1-m^4)[3\mu-2-(3\mu-22)m^4]}{24\cdot 25}, etc$$

One can, as already remarked above, arrange the power sums S_μ in place of the powers of y [and] also in place of the differential quotients of y and y^2 respectively.

By this second demonstration, which is the more elementary, occurs the successive formation according to the formula

$$(7.)\ S_\mu = \frac{\mu-3}{\mu-1}S_{\mu-4} - \frac{m^2}{(\mu-1)(\mu-2)}\frac{\partial^2 S_{\mu-2}}{\partial m^2},$$

where it has been established that $\partial u = m\partial t$, so that thus the differential quotients of y according to u are derived from the equation $\frac{\partial y}{\partial u} = \sqrt{(1-y^4)}$:

thus results the same, what is to remark, from the differential quotients of x in respect to t, if one in these latter according to result of differentiation y is set in the place of of x, then from $(E.)$, if one write there to the right y in place of x. One obtains after this

$$S_4 = \frac{1}{3}p - \frac{1}{6}m^4\frac{\partial^2(x^2)}{\partial t^2}, S_5 = \frac{1}{2}mx + \frac{1}{24}m^5\frac{\partial^2(x^4)}{\partial t^4},$$
$$S_6 = \frac{3}{5}m^2x^2 + \frac{1}{120}m^6\frac{\partial^4(x^2)}{\partial t^4}, etc.,$$

where according to the result of differentiation y is written thoughout in place of x. Since the numerical coefficients in these expressions is exactly the same, as in $(G.)$, thus we obtain from this the following general theorem:

“The sum of the μ th powers S_μ, of the roots of the equation $y = \frac{U}{V}$ yields, if x^μ is expressed in the differential quotients of x or x^2 with respect to t [2], thereafter the constant term with p, the kth differential quotients of x multiplied by m^{k+1} and the kth differential quotients of x^2 multiplied by m^{k+2}, finally according to the result of differentiation y is set in the place of the x.” The law of coefficients of the expansion of x^μ has been given by *Jacobi*(Fundamenta nova page 126 seq.); he shows that the same from the expansion of the powers of t according to the powers of x arises, thus in our case from the expansion of the powers of $\int \frac{\partial x}{\sqrt{(1-x^4)}}$. With this occasion I will indicate a misprint to correct, in the equation of concern: in “Funamenta nova etc.” on page 126 the constant reads in place of the 15th line:

$$-\left(R_{n-2}{}^{(4)}\frac{U^{(2)}}{\prod 4} + R_{n-3}{}^{(6)}\frac{U^{(4)}}{\prod 6} + ... + \frac{U^{(}2n-2)}{\prod 2n}\right),$$

or what according to the same notation there is:

$$-\left(R_{n-2}{}^{(4)} \cdot \frac{1}{3\cdot 4} + R_{n-3}{}^{(6)}\frac{S_1{}^{(2)}}{5\cdot 6} + ...+)\frac{S_n - 2^{(2)}}{(2n-1)2n}\right);$$

there the factors $R_{n-2}{}^{(4)}, R_{n-3}{}^{(6)}$, etc. are ommitted. This constant reduces itself remaining for the lemniscate of a single part and will, as already noted, $= 0$ or$= \frac{1\cdot 5\cdot 9\cdot 13\cdots(4\mu-3)}{3\cdot 7\cdot 11\cdot 15\cdots(4\mu-1)}$. In order to obtain the mentioned formula in the most rapid way, in general one sets the equation

$$(\alpha.)\ \frac{\partial^2\psi}{\partial t^2} = (1-x^4)\frac{\partial^2\psi}{\partial x^2} - 2x^3\frac{\partial\psi}{\partial t}$$

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[2] where x is given by the equation $\frac{\partial x}{\partial t} = \sqrt{(1-x^4)}$.

first $\psi = \dfrac{t^n}{n!}$; then is obtained:

$$\frac{t^{n-2}}{(n-2)!} = (1-x^4)\frac{\partial^2(\dfrac{t^n}{n!})}{\partial x^2} - 2x^3\frac{\partial^2(\dfrac{t^n}{n!})}{\partial x}.$$

This equation becomes differentiated μ times on both sides with reference to x, and acording to the result of diferentiation $x = 0$, whereupon it is remarked, that the differential quotients of $1 - x^4$ disappear with exception of the 0th and 4th and that of x^3 with the exception of the 3rd. It is obtained:

$$\frac{\partial^\mu(\dfrac{t^{n-2}}{(n-2)!})}{\mu!\partial x^\mu} = \frac{\partial^{\mu+2}(\dfrac{t^n}{n!})}{\mu!\partial x^{\mu+2}} - \frac{\partial^{\mu-2}(\dfrac{t^n}{n!})}{(\mu-4)!\partial x^{\mu-2}} - \frac{2\partial^{\mu-2}(\dfrac{t^n}{n!})}{(\mu-3)!\partial x^{\mu-2}},$$

or:

$$(\beta.)\frac{\partial^\mu(\dfrac{t^{n-2}}{(n-2)!})}{(\mu-1!\partial x^\mu} = \mu(\mu+1)\frac{\partial^{\mu+2}(\dfrac{t^n}{n!})}{(\mu+1)!\partial}x^{\mu+2} - \mu(\mu-1)\frac{\partial^{\mu-2}(\dfrac{t^n}{n!})}{(\mu-3)!\partial x^{\mu-2}},$$

for $x = 0$. Thereafter it is supposed in the above formula $\psi = x^\mu$, so that

$$(\gamma.)\frac{\partial^2(x^\mu)}{\partial t^2} = -\mu(\mu+1)x^{\mu+2} + \mu(\mu-1)x^{\mu-2}.$$

Finally now is set

$$x^\mu = (-1)^{\frac{1}{2}(\mu-1)}\sum {P_n}^{(\mu)}\frac{\partial^n x}{\partial t^n}, \text{ or}$$
$$x^\mu = (-1)^{\frac{1}{2}(\mu-2)}\sum {P_n}^{(\mu)}\frac{\partial^n(x^2)}{\partial t^n} + \text{ Constant},$$

depending on whether μ is uneven or even, thus is obtained with respect to $(\gamma.)$

$$(-1)^{\frac{1}{2}}(\mu-1){P_{n-2}}^{(\mu)} = -\mu(\mu+1)(-1)^{\frac{1}{2}(\mu+1)}{P_n}^{(\mu+2)} + \mu(\mu-1)(-1)^{\frac{1}{2}(\mu-3)}{P_n}^{(\mu-2)}$$

respectively,

$$(-1)^{\frac{1}{2}}(\mu-2){P_{n-2}}^{(\mu)} = -\mu(\mu+1)(-1)^{\frac{1}{2}\mu}{P_n}^{(\mu+2)} + \mu(\mu-1)(-1)^{\frac{1}{2}(\mu-4)}{P_n}^{(\mu-2)}$$

then in both cases:

$$(\delta.){P_{n-2}}^{(\mu)} = \mu(\mu+1){P_n}^{(\mu+2)} - \mu(\mu-1){P_n}^{(\mu-2)}.$$

This recursion equation satisfies according to $(\beta.)$

$$(\epsilon.){P_n}^{(\mu)} = \frac{\partial^{\mu-2}(\dfrac{t^{n+2}}{n+1!})}{(\mu-1)!\partial x^\mu}, or = \frac{\partial^\mu(\dfrac{t^{n+2}}{(n+2)!})}{(\mu-1)!\partial x^\mu} \text{ for } x = 0,$$

and since this value is true, if $\mu = 1, \mu = 2, 3$ and 4, whereas generally n remains[3], thus also it becomes for all values of μ and n, and obtains:

$$(8.)\ x^{\mu} = (-1)^{\frac{1}{2}(\mu-1)} \sum_{n=0} \frac{1}{(\mu-1)!(n+1)!} \frac{\partial^n x}{\partial t^n} \Big(\frac{\partial^{\mu(t^n+1)}}{\partial x^{\mu}}\Big)_{x=0}$$

or

$$= (-1)^{\frac{1}{2}(\mu-2)} \sum_{n=0} \frac{1}{(\mu-1)!(n+2)!} \frac{\partial^n (x^2)}{\partial t^n} \Big(\frac{\partial^{\mu}(t^{n+2})}{\partial x^{\mu}}\Big)_{x=0} + C,$$

depending on whether μ is uneven or even; furthermore $C = 0$ or $C = \frac{1 \cdot 5 \cdot 9 \cdots (\mu-3)}{3 \cdot 7 \cdot 11 \cdots (\mu-1)}$, depending on whether $\mu \equiv 2$ or $\equiv 0 (mod. 4)$. to remark, that in this formula (8.) those coefficients disappear automatically, for which $n+1$ and $n+2$ are not $\equiv \mu (mod. 4)$respectively. For the μth power sum it results now from this through the employment of the theorems assembled above:

$$(9.)\ S_{\mu} = \sum x^{\mu} = (-1)^{\frac{1}{2}(\mu-\sigma)} \sum \frac{m^{n+\sigma}}{(m-1)!(n+\sigma)!} \frac{\partial^{\mu}(t^{n+\sigma})}{\partial x^{\mu}} \frac{\partial^n (y^{\sigma})}{\partial u^n} + Cp,$$

where $\sigma = 1$ or $= 2$, depending on whether μ is uneven or even, $\frac{\partial y}{\partial u} = \sqrt{(1-y^4)}, C = 0$ for $\mu \equiv 1, 2, 3 (mod. 4)$, in contrast $C = \frac{1 \cdot 5 \cdot 9 \cdots (\mu-3)}{3 \cdot 7 \cdot 11 \cdots (\mu-1)}$ for $\mu \equiv 0 (mod. 4)$, and where according to the result of differentiation to the right x must become set $= 0$.—The sums of the negative powers $\sum x^{-\mu}$ can also yet be obtained, if $\frac{1}{y}$ is written in place of y, then since y passes over into $\frac{1}{y}$, if $\frac{1}{x}$ is set in place of x, then $\sum x^{-\mu}$ is obtained from the values of $\sum x^{\mu}$, if in the latter $\frac{1}{y}$ is set in place of y.—Generally, from the power sums S_{μ}, which yet contains the variable y, results the special power sums of the numerators of the multiplication formulas $U = 0$, which we denote byS'_{μ}, if y is set $= 0$; however we set $y = 0$, thus in (9.)$\frac{\partial^n (y^{\sigma})}{\partial u^n}$ can be written in place of[auch] $\frac{\partial^n (y^{\sigma})}{\partial t^n}$, if only then according to the result of differentiation$x$ will be set $= 0$. S'_{μ} always vanishes, if $\mu \equiv 1, 2,$ or $3 (mod. 4)$; in contrast for the values of μ divisible by 4 one obtains:

$$(10.)\ S_{\mu} = -\sum \frac{m^{n+2}}{(\mu-1)!(n+2)!} \frac{\partial^{\mu}(t^n+2)}{\partial x^{\mu}} \frac{\partial^n (x^2)}{\partial t^n} \{x = t = 0\}$$
$$+ \frac{1 \cdot 5 \cdot 9 \cdots (\mu-3)}{3 \cdot 7 \cdot 11 \cdots (\mu-1)} p.$$

The vanishing of all the sums in those terms, for which n is not $\equiv 2 (mod. 4)$, so that only those powers of m remain, whose exponents are divisible by 4; one can also set:

$$(10'.)\ S_4\mu = -\sum_{n=1}^{n=\mu} \frac{m^{4n}}{(4\mu-1)!(4n)!} \frac{\partial^4\mu(t^4 n)}{\partial x^4\mu} \frac{\partial^(4n-2)(x^2)}{\partial t^(4n-2)} + \frac{1 \cdot 5 \cdots (4\mu-3)}{3 \cdot 7 \cdots (4\mu-1)} p.$$

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[3] Since for $\mu = 1, \mu = 2$ depending on the values of n on both sides of the forumula $1, 0, 0, 0, etc; for \mu = 3$ appears $0, 0, \frac{1}{2}, 0, 0....$, and for $\mu = 4 : 0, 0, \frac{1}{6}, 0, 0, ...$

At the same time 4μ has been written in place of μ; because it is only about values of μ divisible by 4. These formulas (10.) or (10′.) give various power sums of the roots of the equation

$$x^{p-1} + B_1 x^{p-5} + B_2 x^{p-9} + ... + A_x^4 + m = 0,$$

and if $S'{}_4\mu = 4 \cdot T_\mu$, evidently T_μ constitutes the sum of the μth power of the following equation

$$z^{\frac{1}{4}} + B_1 z^{\frac{1}{4}(p-1)-1} + ... + A_1 z + m = 0,$$

which from the above by substitution emerges $x^4 = z$; these power sums of the roots of the equation in z is furthermore

$$T_\mu = \frac{1}{4} S'{}_4\mu = \sum \varphi(\frac{rw}{m})^{4\mu},$$

where r represents the sum of a quarter system of residues; therefrom it is found:

$$\begin{aligned}
&\sum \varphi(\frac{rw}{m})^4 = T_1 = \frac{p - m^4}{12},\\
&\sum \varphi(\frac{rw}{m})^8 = T_2 = \frac{1}{4} \cdot \frac{1 \cdot 5}{3 \cdot 7} p - \frac{1}{15} m^4 + \frac{1}{4} \cdot \frac{1}{35} m^8,\\
&\sum \varphi(\frac{rw}{m})^{12} = T_3 = \frac{1}{4} \cdot \frac{1 \cdot 5 \cdot 9}{3 \cdot 7 \cdot 11} p - \frac{11 m^4}{3 \cdot 4 \cdot 25} + \frac{3 m^8}{2 \cdot 7 \cdot 25} - \frac{m^{12}}{2 \cdot 3 \cdot 11 \cdot 25} etc.
\end{aligned}$$

From the power sums T_1, T_2, etc the coefficients B_1, B_2, etc can now be calculated, and indeed according to the acknowledged formula *Newton*

$$\begin{gathered}
B_1 = -T_1, 2B_2 = -T_2 - B_1 T_1, 3B_3 = -T_3 - B_1 T_2 - B_2 T_1,\\
4B_4 = -T_4 - B_1 T_3 - B_2 T_2 - B_3 T_1, etc.
\end{gathered}$$

From this it is found e.g.

$$\begin{aligned}
B_1 &= \frac{-p + m^4}{12},\\
B_2 &= \frac{p^2}{2 \cdot 144} - \frac{5}{14 \cdot 12} p + m^4 (\frac{-p}{144} + \frac{1}{30}) - \frac{m^8}{8 \cdot 35 \cdot 36}\\
&= \frac{1}{56\dot{1}80} (-m^8 - 70 m^4 p + 35 p^2 + 336 m^4 - 300 p), etc
\end{aligned}$$

In a simlar way the coefficients A_1, A_1 etc., are found:

$$\begin{aligned}
A_1 &= \frac{m}{60} (-m^4 - 5p + 6),\\
A_2 &= \frac{m}{56 \cdot 180 (-m^8 + 14 m^4 p + 35 p^2 - 84 m^4 - 384 p + 420)}, etc
\end{aligned}$$

If it is set, what is afforded,

$$B_\mu = \alpha_\mu + \beta_\mu m^4 + \gamma_\mu m^8 + ... + \lambda_\mu m^{4\mu},$$

then it is obtained from the combination of (10′.) with the *Newtonian* formula e.g.

$$\mu\alpha_\mu = -\frac{1}{4}p\left[\frac{1\cdot 5\cdots(4\mu-3)}{3\cdot 7\cdots(4\mu-1)}+\frac{1\cdot 5\cdots(4\mu-7)}{3\cdot 7\cdots(4\mu-5)}\alpha_1+....+\frac{1\cdot 5}{3\cdot 7}\alpha_{\mu-2}+\frac{1}{3}\alpha_{\mu-1}\right],$$

etc. for β_μ, γ_μ etc. similar formula.

We put forward yet the following remarks concerning the coefficients of the numerators and denominators along with the multiplication formulas for the lemniscate:

a) The general coefficient B_μ of the denominators is regarded as a whole function of both variables $m^4$and p of the μth order, in which the constant part is lacking; the general coefficient A_μ of the numerators is regarded as equal to the m multiple of a whole function to the μth order of both variables m^4 and p; the coefficients of the powers and products of m^4 and p in these whole functions are purely *numerical* and *real*, and depend neither on m nor p, its law remains yet unknown and can also in a way, as here adopted, become found with difficulty; one could indeed insert the values of $T_\mu = \frac{1}{4}S'{}_4\mu$ according to (10′.) in the independent expression of the coefficients by the power sum,

$$B_\mu = \sum\frac{(-1)^{\alpha+\beta+\gamma+....}{T_1}^{\alpha}{T_2}^{\beta}{T_3}^{\gamma}....}{1^\alpha\cdot 2^\beta\cdot 3^\gamma....\alpha!\beta!\gamma!....}$$

$$\{\text{on the condition that } \alpha+2\beta+3\gamma\cdots=n\}$$

thus in obtaining a law of coefficients however, its nature would be discovered as more hidden.

b) Although the norm p in the coefficients rises up to the μth degree, thus the refined power sum is comprised from its roots however p merely *linear* and also not multiplied by the powers of m.

c) The coefficients, owing to their form, terminate automatically, once μ is greater than $\frac{1}{4}(p-1)$. After this, in combination with *b*) one can calculate any number of coefficients independently from the previous treatments and knowledge of power sums, while one can establish numerical coefficients occuring in [Ihnen] as unknowns and thus as many equations can be built, as there are unknowns.

$B-1 = \frac{1}{12}(m^4-p)$ and $A_1 = \frac{1}{60}m(-m^4-5p+6)$ e.g. vanishing for $m=1$, and $B-1$ will be $=-1+2i, A_1 = 1$ for $m=-1+2i, p=5$; from this alone both of these coefficients could have been founded; then $B-1$ is set $=\alpha m^4+\beta p, A_1 = m(\gamma m^4+\delta p+\epsilon)$, thus obtaining the five equations

$$\begin{aligned}\alpha+\beta=0,&(-1+2i)^4\alpha+5\beta=-1+2i,\\ \gamma+\delta+\epsilon=0,&(-1+2i)^5\gamma+5(-1+2i)\delta+(-1+2i)\epsilon=1,\\ &(-1-2i)^5\gamma+5(-1-2i)\delta+(-1-2i)\epsilon=1,\end{aligned}$$

whereby the 5 unknowns $\alpha, \beta, \gamma, \delta, \epsilon$ are completely determined. Both later coefficients must, in order to obtain an adequate number of equations, as already remarked, yet add the quality of *b*). Also one advantageously makes use of the relationship here between the coefficients of the numerators and denominators of (§.1.) *d*) As just demonstrated, is for a two-termed prime number m the values of A_μ and B_μ divisible by m. *e*) In B_μ the coefficient of p, $= \frac{-1}{4\mu}\frac{1 \cdot 5 \cdot 9 \cdots (4\mu - 3)}{3 \cdot 7 \cdot 11 \cdots (4\mu - 1)}$ the coefficients of p^μ, $= \frac{(-1)^\mu}{\mu! 12^\mu}$, that of $p^{\mu-1}, = \frac{5}{14}\frac{(-1)^{\mu-1}}{(\mu-2)! 12^{\mu-1}}$. *f*) If in the coefficients p is replaced with an *arbitrary* magnitude, thus remains even now the correct(due) multiplication formula; however, then the numerator and denominator are only dissolved if the norm p is of m. I give this attempt, thus it may also be incomplete, since the contribution can be of interest due to the great difficulties of the subjects of each. A practical formula is yet obtained, if (9.) is differentiated on both sides with respect to t:

$$(11.)\ \sum (x^{\mu-1} \cdot \sqrt(1 - x^4)) = \frac{(-1)^{\frac{1}{2}(\mu-\sigma)}}{\mu!} \sum \left\{ \frac{m^{n+\sigma+1}}{(n+\sigma)!} (\frac{\partial^\mu (t^{n+\sigma})}{\partial x^\mu})_{x=0} \cdot \frac{\partial^{n+1}(y^\sigma)}{\partial u^{n+1}} \right\}.$$

5.

The method of the forgoing paragraphs for the determination of the symmetrical functions of the roots of such equations, which of the specified differential equation suffices, extends much farther than we have indicated here; it can be made use of in more problems about elliptical functions and still in a manifold of other formulas of integral algebratic functions. Here we will, with its aid, promptly determine yet only the symmetrical combination of the roots of those equations, which are found via the transformation of elliptical functions for any modulus.[4]

$$I.\ \text{If } \frac{\partial x}{\partial t} = \sqrt{((1.x^2)(1 - k^2x^2))},\ \text{thus} \frac{\partial^2 x}{\partial t^2} = -(1 + k^2)x + 2k^2x^3,$$

Thus the general equation is had

$$(1.)\ \frac{\partial^2 \psi}{\partial x^2}(1 - (1 + k^2)x^2 + k^2x^4) + \frac{\partial \psi}{\partial x}(-(1 + k^2) + 2k^2x^3),$$

whre ψ is any one function of x or t. If one first sets $\psi = x^\mu$, thereafter $\psi = t^h$, thus is obtained:

$$(2.)\ \frac{\partial^2 (x^\beta)}{\partial t^2} = \mu(\mu + 1)\alpha x^{\mu+2} - \mu^2 \beta x^\mu + \mu(\mu - 1)x^{\mu-2},$$

$$(3.)\ h(h - 1)t^{h-2} = \frac{\partial^2 (t^h)}{\partial x^2}(1 - \beta x^2 + \alpha x^4) + \frac{\partial (t^h)}{\partial x}(-\beta x + 2\alpha x^3),$$

where the to save time we have set $k^2 = \alpha, 1 + k^2 = \beta$. If the latter equation (3.) is differentiated μ times with respect to x and set according to the result of differentiation $x = 0$, thus appears

$$\begin{aligned} h(h-1)\frac{\partial^\mu (t^{h-2})}{\mu! \partial x^\mu} = \frac{\partial^{\mu+2}(t^h)}{\mu! \partial x^{\mu+2}} - \beta \frac{\partial^\mu (t^h)}{(\mu - 2)! \partial x^\mu} + \alpha \frac{\partial^\mu (t^h)}{(\mu - 4)! \partial x^{\mu-2}} \\ - \beta \frac{\partial^\mu (t^h)}{(\mu - 1)! \partial x^\mu} + 2\alpha \frac{\partial^{\mu-2}(t^h)}{(\mu - 3)! \partial x^\mu - 2} \end{aligned}$$

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[4] Especially if one compares also the “Notices sur les fenct. ellipt.” from *Jacobi* in the 3rd and 4th volume of *Crelles* jounal.

or, if it is reduced,

$$(4.)\ \frac{\partial^{\mu}(\dfrac{t^{h-2}}{(h-2)!})}{(\mu-1)!\partial x^{\mu}} =$$

$$\mu(\mu+1)\frac{\partial^{\mu+2}(\dfrac{t^h}{h!})}{(\mu+1)!\partial x^{\mu+2}} - \mu^2\beta\frac{\partial^{\mu}(\dfrac{t^h}{h!})}{(\mu-1)!\partial x^{\mu}} + \mu(\mu-1)\alpha\frac{\partial^{\mu-2}(\dfrac{t^h}{h!})}{(\mu-3)!\partial x^{\mu-2}};$$

thus will $\dfrac{\partial^{\mu}(\dfrac{t^h}{h!})}{(\mu-1)!\partial x^{\mu}} = R_h{}^{(\mu)}$ for $x = 0$, then it is had that

$$(4'.) R_{h-2}{}^{(\mu)} = \mu(\mu+1)R_h{}^{(\mu+2)} - \mu^2\beta R_h{}^{(\mu)} + \mu(\mu-1)\alpha R_h{}^{(\mu-2)}.$$

II. In virtue of (2.) the following expansion can become assumed:

$$\alpha^{\frac{1}{2}(\mu-\sigma)}x^{\mu} = \sum P_h{}^{(\mu)}\frac{\partial^h(x^{\sigma})}{\partial t^h} - Const.,$$

where $\sigma = 1$ or $= 2$, depending on whether μ is uneven or even; e.g. for $\mu = 1$ is x, thus $P_h{}^{(1)}$ for $h = 0$ and $= 0$ for $h > 0$; for $\mu = 3, \alpha x^3 = \frac{1}{2}\beta x + \frac{1}{2}\dfrac{\partial^2 x}{\partial t^2}$, thus the following values of h is assumed for the values of $P_h{}^{(3)}$: $\frac{1}{2}\beta, 0, \frac{1}{2}, 0, 0, 0, etc.$; for $\mu = 4$ will

$$\alpha x^4 = \frac{2}{3}\beta x^2 + \frac{1}{6}\frac{\partial^2(x^2)}{\partial t^2} - \frac{1}{3},$$

thus the values of $P_h{}^{(4)}$ are the following: $\frac{2}{3}\beta, 0, \frac{1}{6}, 0, 0, 0, etc.$. If the above expansion is set in equation (2.), thus comes

$$(5.)\ P_{h-2}{}^{(\mu)} = \mu(\mu+1)P_h{}^{(\mu+2)} - \mu^2\beta P_h{}^{(\mu)} + \mu(\mu-1)\alpha P_h{}^{(\mu-2)}.$$

This relation is satisfied according to (4′.):

$$(6.)\ P_h{}^{(\mu)} = R_{h+\sigma}{}^{(\mu)},$$

and when this correpondence the initial value of μ, while h generally remains, can become proved, thus must is generally occur, because $P^{(\mu+2)}$ will be calculated exactly as from P with the above indices μ and $\mu - 2$, as $R^{(\mu+2)}$ from R with the above indices μ and $\mu - 2$. This correspondence is found in fact for $\mu = 1, 2, 3, 4$[Statt], then $P_{h+1}{}^{(1)}, P_{h+2}{}^{(2)}, P_{h+1}{}^{(3)}, P_{h+2}{}^{(4)}$ are assumed for $h = 0, 1, 2, 3, ...$, in [inf.] the values respectively:

$$\begin{aligned} R_{h+1}{}^{(1)} &= 1, 0, 0, 0, 0, ..., \\ R_{h+2}{}^{(2)} &= 1, 0, 0, 0, 0, ..., \\ R_{h+1}{}^{(3)} &= \frac{1}{2}\beta, 0, \frac{1}{2}, 0, 0, ..., \\ R_{h+2}{}^{(4)} &= \frac{2}{3}\beta, 0, \frac{1}{6}, 0, 0, ..., \end{aligned}$$

and these values add up exactly with the previous initial value of $P_h{}^{(\mu)}$ for $\mu = 1, 2, 3,$ and 4; thus (6.) generally is so. According to this we obtain:

$$(7.)\ k^{\mu-\sigma}x^{\mu} = \sum R_{h+\sigma}{}^{(\mu)}\frac{\partial^h(x^{\sigma})}{\partial t^h} - Const.$$

where

$$R_{h+\sigma}{}^{(\mu)} = \frac{1}{(h+\sigma)!(\mu-1)!}(\frac{\partial^\mu(t^{h+\sigma})}{\partial x^\mu})_{x=0}.$$

The constant will be determined from the special case where $x = 0, t = 0$; it is found

$$Const. = \sum R_{h+\sigma}{}^{(\mu)} \cdot (\frac{\partial^h(x^\sigma)}{\partial t^h})_{t=0}.$$

III. For each uneven number n, as noted by *Jacobi* and Abel, can indicate multiple values belonging to λ and N, so that

$$\frac{\partial y}{\partial t} = N \cdot \sqrt{((1-y^2)(1-\lambda^2 y^2))} = N \cdot \frac{\partial y}{\partial u}, \partial = N \cdot \partial t$$

will, while y is a non-integral rational function of x, which vanishes together with x, and whose numerator and denominator rises to the nth and $n-1$th degree respectively. Let

$$y = \frac{U}{V} = \frac{k}{N\lambda}\frac{x^n + Bx^{n-2} + B_2x^{n-4} + ...}{x^{n-1} + A_1x^{n-3} + A_2x^{n-5} + ...};$$

that y must assume this form, is yielded immediately by the insertion in the differential equation and has been remarked by *Jacobi* in multiple locations. The roots of the equation

$$U - Vy = 0, i.e.$$

$$x^n - \frac{N\lambda}{k}y\,.^{n-1} + Bx^{n-2} - etc. = 0$$

are denoted by

$$x_1, x_2, x_3, ...x_n,$$

and the sum

$$F(x_1) + F(x_2) + F(x_3) + ... + F(x_n) \text{ is denoted by} \sum F(x).$$

The sum of the magnitudes $x_1, x_2, x_3, ...x_n = \frac{N\lambda}{k}y$, the sum of its combinations in pairs $= B$, thus the sum of its squares$= \frac{N^2\lambda^2}{k^2}y^2 - 2B$. From this emerges two equations, which can shortenened in the following only:

$$(8.)\ \sum_x x^\sigma = \frac{N^\sigma\lambda^\sigma}{k^\sigma}y^\sigma - c,$$

where $\sigma = 1$ and $= 2$, while according to these two cases $c = 0$ or $= 2B$ respectively. The equation (8.) is differentiated on both sides h times with respect to t, where this is to be noted, that $\partial u^h = N^h\partial t^h$, thus appears:

$$(9.)\ \sum_x \frac{\partial^h(x^\sigma)}{\partial t^h} = \frac{N^{h+\sigma}\lambda^\sigma}{k^\sigma}\frac{\partial^h(y^\sigma)}{\partial u^h};$$

for $h = 0$ enters the equation (8.) in the place of (9.).

IV. If now the series of values $X_1, x_2, x_3, ...x_n$ will have been set in (7.) for x, its sum, and finally for

$$\sum_x \frac{\partial^h(x^\sigma)}{\partial t^h}$$

the values inserted from (8.) and (9.), thus appears:

$$(10.)\ k^{\mu-\sigma}\sum_x x^\mu = \sum_{h=0}^{h=\mu-\sigma} R_{h+\sigma}{}^{(\mu)}\frac{N^{h+\sigma}\lambda^\sigma}{k^\sigma}\frac{\partial^h(y^\sigma)}{\partial u^h} - n\sum_{h=0}^{h=\mu-\sigma} R_{h+\sigma}{}^{(\mu)}(\frac{\partial^h(x^\sigma)}{\partial t^h})_{t=0} - cR_\sigma{}^{(\mu)};$$

σ is, as already remarked, $= 1$ or $= 2$, depending on whether μ is uneven or even; the significance of the coefficients $R_{h+\sigma}{}^{(\mu)}$ is indicated above, and x,y becomes properly determined in with respect to t and u by the differential equation:

$$\frac{\partial x}{\partial t} = \sqrt{((1-x^2)(1-k^2x^2))} \text{ and } \frac{\partial y}{\partial u} = \sqrt{((1-y^2)(1-\lambda^2y^2))}.$$

Since according to the latter the complete differential quotient $\frac{\partial^h(y^\sigma)}{\partial u^\sigma}$ can be converted into powers y, thus the formula (10.) is given the μth power sum of the roots of the equation $U - Vy = 0$ expressed completely in whole functions of y. From the power sum all other symmetrical combinations of these roots can be derived: the coefficients of the numerators and denominators U and V of the transformation formulas can thus be similarly found, and indeed with the aid of the four magnintudes

$$k, \lambda, N \text{ and } B.$$

This latter complicated problem is the first to have been solved by *Jacobi* in a completely different way(Volume 4 of *Crelles* Journal). For the case of the multiplication is to set n^2 in place of n; further $N = n, \lambda = k$ and$B = 0$, thus the symmetrical functions become purely expressed in n and k without the aid of other magnitudes. With the aid of theorems from *Lagrange* concerning the inversion of series, the equation (10.) can take multiple other forms.

V. If n is a prime number, thus in (9.) its value $n + 1$ can be set in place of λ and the accompanying value inplace of N. If the left and right of all these values of λ and N are added, thus the analogous series for the multiplication is had, which in turn can be immediately found from (9.), if it is set that $\lambda = k, N = n$. Further now $t = 0$ and u is taken to $= 0$, thus to the left are pure functions of k, while to the right stand symmetrical combinations of the magnitudes λ and N. Finally in this way various symmetrical combinations of the transformed modulus and the accompanying multipliers expressed in k. The coefficients of the modular equation are not obtained thereby however, because it is more complicated to isolate the values of N from those of λ. In contrast, all coefficients can be eliminated from the *Newtonian* formula and any number of equations can be constructed. ν is placed to mean formed equations, where ν must become > 3. If now in these ν equations the value of the power sum of the roots of the equation $U = 0$ one, which emerge from (10.), if there to the right from the result of differentiation y is set $= 0$. The ν equations obtain then out of k still the three magnitudes λ, N and B. If between them N and B is eliminated, thus is the modular equation the greater common divisor the emerging out of this $\nu - 2$ equations; the greatest common denominator of the $\nu - 2$ equations emerging from the elimination of λ and B is an equation between N and k, and the greatest common denominator of those, which arises through the elimination of λ and N, is an equation between B and k.

We admit, that this method to find the modular equation, if it should not become illusory for a *general* value of n, certainly requires yet a significant development.

In October 1845.