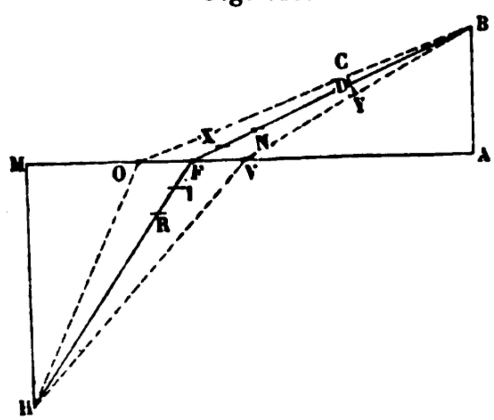
Demonstration from the Letter to M. de ···*

1664

Given line AFM (*fig.* 105), which represents the separation of the two different media; let air be on the side of B and water on the side of H. The ray of light, which must go from point B, which is in the air, towards point F, where the medium of water begins, breaks and goes towards H, moving towards the perpendicular, according to common everyday experiments.

Fig. 105.



M. Descartes determines this point H in such a way, that by drawing a perpendicular BA from B on the line AFM, he makes line AF be to line FM as the resistance in one of the media is to the other, even though he believes, contrary to my understanding, that the resistance is greater in air than in water.

Therefore let the greater resistance be represented by line AF and the lesser by line FM, making line AF consequently greater than FM. Let there

*Cf. the "Synthesis for Refractions," included as the last section of Fermat's writing on Minima and Maxima, and the demonstration by Huyghens in his *Treatise on Light.*

be erected from point M, the perpendicular MH which is cut at H by the circle with center F and radius FB, so that lines BF and FH will be equal: I say that the radius BF, immediately after being broken by the encounter with the water, will go towards H.

For, since, by my principle, nature always acts along the shortest paths, if I prove that in traversing the two lines BF and FH, it employs less time than by traversing any other point on the line AM, I will have proven the truth of the proposition.

Now, since I have assumed in advance that motion through air is easier and consequently faster, the motion from B to F will be made in less time than that from F to H, and, to set the true proportion, we must make

as AF to FM (which are the measures of the resistances), so is BF to FD,

and the two lines DF and FH will be the measures of the time which will be used from B to F and from F to H. That is, line DF will be the measure of the motion along BF, which is quicker, and the line FH will be the measure of the motion along FH, which is slower, and this, according to the proportion of BF to FD, or of HF – which is equal to BF – to the same FD.

Therefore if I prove that, for whatever point you take on the two sides of DF, the sum of these two lines DF, FH is always smaller than any other two lines moving in the same direction, I will have that which I was seeking.

Therefore let there first be point O on the side of M. By joining lines BO and OH, and making

as BF is to DF, so is BO to CO,

I must prove that the sum of the two lines CO and OH is greater than that of DF and FH. And, similarly, by taking a point such as V, on the side of A, I must also prove that by joining the two straight lines BV and VH, and making

as BF is to DF, so is BV to YV,

the sum of the two lines VY and VH is greater than that of the two lines DF and FH.

To get there, I make

as BF to AF, so is FO to FR

and

as the same BF is to FM, so is FO to FI.

Since BF is greater than AF, therefore FO is greater than FR and, since AF is greater than FM, FR is also greater than FI, and it similarly appears that

FR is to FI as AF is to FM.

For, since, by the construction

as AF is to FB, so is FR to FO,

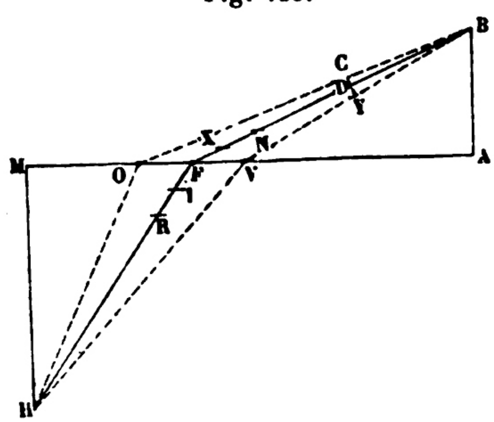
and

as FB is to FM, so is FO to FI,

therefore, *ex æquo*,

as AF is to FM, so is FR to FI.

Fig. 105.



I say, therefore, that the two lines CO and OH are greater than the two lines DF and FH. For, by Euclid, in the obtuse triangle FHO, the sum of the two squares of HF and FO is equal to the sum of the square of HO and the rectangle MFO taken twice. Now, since we have made

as BF or FH is to FM, so is FO to FI,

therefore the rectangle under the extremes HFI is equal to the rectangle under the means MFO, and the rectangle HFI taken twice is equal to the rectangle MFO taken twice: we therefore have the sum of the two squares HF and FO equal to the sum of square HO and rectangle HFI taken twice. But the rectangle HFI taken twice is equal to the rectangle HIF taken twice with double the square of IF; and the square HF, according to the same Euclid, is equal to the rectangle HIF taken twice with the two squares HI and IF: we therefore have, on one side, the square HI, the square IF, the rectangle HIF taken twice and the square FO equal to the square HO, to the rectangle HIF taken twice and to the square FI taken twice. Remove HIF taken twice and the square FI from both sides: there remains, on one side, the square HI with the square FO equal to two squares HO and IF. But the square FO is greater than the square FI, since, by the construction, FO is greater than FI: therefore square HO is greater than the square HI, hence the line HO is greater than the line HI.

If I then prove that the line CO is greater than the two lines DF and FI, there will remain proven that the two CO and OH are greater than the three DF, FI, and IH, or the two DF and FH: I therefore prove what is required.

According to Euclid, in the obtuse triangle BFO, the square of BO is equal to the sum of the squares on BF and FO and twice the rectangle AFO; but, since we have made

FO to FR as BF is to FA

by construction, therefore the rectangle under BF and FR is equal to the rectangle AFO, and consequently the square of BO is equal to the squares on BF and FO and to the rectangle under BF, FR taken twice. But the square on FO is bigger than that on FR, since the line FO has been proven to be greater than the line FR: therefore, if you substitute the square on FR in place of the that on FO, the square BO will be greater than the two squares BF, FR and the rectangle BFR taken twice. But these last sums are equal, by Euclid, to the squares of the two lines BF and FR taken as one: therefore the line BO is greater than the sum of the two lines BF and FR. But we have proved that

RF is to IF as AF is to FM, that is as BF is to FD,

which is the measure of the diversity of motions: therefore,

as the sum of the two former BF and FR
is to the sum of the two latter DF and FI,
so is BF to FD.

Now

BO is to OC as BF is to FD:

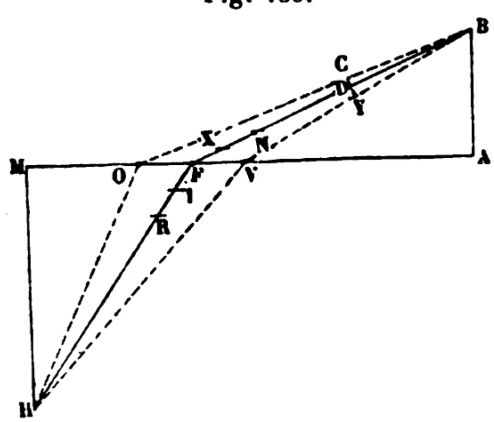
therefore

as BO is to OC,
so is the sum of the two lines BF and FR
to the sum of the two lines DF and FI.

But we have proved that the line BO is greater than the sum of the two lines BF and FR: it is therefore true that the line CO is greater than the sum of the two lines DF and FI, which was the second thing to be proven.

Therefore there is no other point on the side of M through which the ray may pass without taking more time than by going through point F. The same remains to be proven about point V.

Fig. 105.



If we proceed, as above,

as BF is to FA, so is FV to FN

and

as the same BF is to FM, so is FV to FX,
NF will be to XF as AF to FM, that is as BF is to FD,

by the preceding proof, and both of the two lines NF and XF will be smaller than VF, by what has come before.

It is necessary to prove that the sum of the two lines YV and V is greater than the sum of the two lines DF and FH.

First, I consider that, by Euclid, in the obtuse triangle VFH, the sum of the squares on HF and FV and twice the rectangle MFV is equal to the square VH; but, since, by construction, it has been made that

as BF is to FM, so is FV to FX,

therefore the rectangle BFX or HFX (since BF and FH are equal) is equal to the rectangle MFV: we therefore have, on one side, the sum of the squares on HF and FV and of twice the rectangle HFX equal to the square on HV. But the square on FX is less than the square on FV: therefore the sum of the squares HF, FX and twice the rectangle HFX is less than the square HV. Yet this sum is equal to the square made on the two lines HF and FX as one, by Euclid: therefore the sum of the two lines HF and FX is less than HV, ad HV is greater than that of the two lines HF and FX.

Therefore, if I prove that the lie YV is greater than the line DX, there will remain to be proven that the sum of YV and HV is greater than the sum of DX, XF, FH that is the sum of DF, FH.

To make this last proof, I consider the obtuse triangle BVF in which, according to Euclid, the two squares BF, FV are equal to the square BV and the rectangle AFV taken twice; now, since, by construction, we have made

BF to FA as VF to FN,

therefore rectangle BFN is equal to rectangle AFV. Hence, the sum of the two squares BF and FV is equal to the sum of square BV and twice the rectangle BFN. Yet the rectangle BFN taken twice is equal to the rectangle BNF and twice square FN: therefore the sum of the two squares BF and FV is equal to the sum of the square BV, the rectangle BNF taken twice, and the square of FN taken twice. Yet the square BF is, by Euclid, equal to the square BN, the square NF and the rectangle BNF taken twice: we therefore have the sum of squares BN, NF, FV and rectangle BNF taken twice and of the square of FN taken twice. Remove rectangle BNF and square NF from each side: it therefore remains that the square of BN and the square FV will be equal to the squares BV and FN. Yet square FV is greater than the square of FN, by construction: therefore the square BV is greater than that on BN, and therefore the line BV is greater than line BN.

We have proved that

as BF is to FD, so is NF to FX:

therefore

as line BF is to FN, so is DF to FX,

and, by the conversion of ratios,

as BF is to BN, so will DF be to DX,

and

as BF is to DF, so will BN be to DX.

But we have made

BF to DF as BV to YV

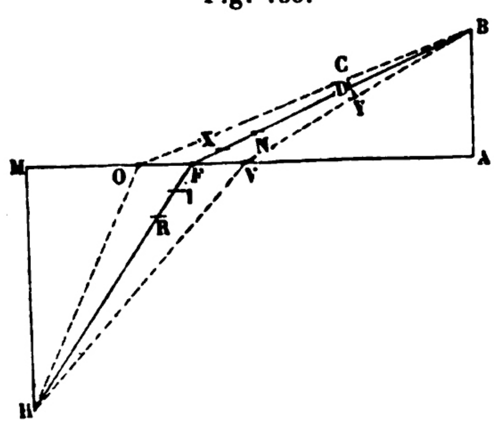
therefore

as BV is to YV, so will BN be to DX.

But we have proved that BV is greater than BN: therefore YV will be greater than it more than DX is.

Yet it has already been proven that VH is greater than the two lines HF and FX: therefore it is plainly proven that the two lines YV and VH are greater than the three DX, XF, FH, or than the two DF and FH, and thus the demonstration is complete.

Fig. 105.



From this it follows that by proposing my principle, that nature acts always by the shortest paths, the assumption M. Descartes is false, when he says that the movement of light is made more easily in water and other dense bodies than in the air or and other rare bodies.

For, if this assumption of M. Descartes were true and if you were to imagine that, in my figure, air were on the side of H and water on the side of B, it would follow, by transposing the demonstration, that the ray which leaves point H and encounters the water at point F, would deflect towards B, because, motion in air being slower according to the assumption of M. Descartes, it would be measured by the line HF, and the one going through water would be measured by FD, as being faster, such that, the two lines HF and FD being the shortest, refraction would be made towards B, that is to say that the ray would be drawn away from the perpendicular, which is absurd and contrary to experiment.

If the situation of the two points B and H changes along the two lines BF and FH, each extended as far as you like, the demonstration will hold, and you will see it yourself.

I do not add the analysis, for, beyond its being long and awkward, it should be sufficient that the account you have just read is short and purely geometrical.

From all this it follows that, given points B and F, or H and F, the problem can be solved easily by means of areas. But, when one is given two points, such as B and H, and desires to search from them the point of refraction on the line or the plane which separates the two media, in this case the problem is solid, and can only be constructed by using parabolas, hyperbolas and ellipses. But, since this construction is not even difficult for a mediocre geometer, provided he remains in agreement with the foundation and the proportion with which he must work and that I have already explained to you, I have no doubt that you would have found it first, you, Sir, who are so much above the common.

Aside from the fact of dealing here, in the question that you have asked me, with nothing else but learning the pathways that nature teaches us to follow, and that this great worker has no need of our instruments and of our machines, I have already fulfilled my task.