Introduction to Surface Loci

To my Friend M. de Carcavi

Pierre de Fermat

Translated from French by Jason Ross

To complete the *Introduction to Plane and Solid Loci*,[1] there remains the treatment of *surface loci.* The ancients only indicated this subject, but neither taught the general rules nor did they even give any famous examples. Or if they did, they have been buried long ago in the monuments of ancient Geometry where so many precious discoveries have been abandoned, defenseless, to the insects, and often annihilated without leaving a trace.

This theory, however, is susceptible of treatment by a general method, as this short dissertation will show; later, if we have the leisure, we will further clarify each of the geometric discoveries which we have until now so briefly made known.

The characteristics that we have sought and demonstrated in lines as loci can also be studied for plane, spherical, conical, and cylindrical surfaces, or by those of arbitrary conoids and spheroids,[2] if we first establish the lemmas constituting each of these loci.

Let us therefore pose the following lemma for plane surface loci:

1. *If a given surface is cut by as many planes as you wish, and the intersection of this surface with each of the planes is always a straight line, then the surface in question will be a plane.*

For loci on spherical surfaces:

2. *If a given surface is cut by an arbitrary number of planes, and the intersection of the surface with each of the planes is always a circle, then the surface in question will be a sphere.*

For loci on spheroidal surfaces:

3. *If a given surface is cut by an arbitrary number of planes, and the intersection of the surface with each of the intersecting planes is either a circle or an ellipse, but never another curve, then the surface in question will be a spheroid.*

For loci on parabolic conoid or hyperbolic surfaces:

4. *If a given surface is cut by an arbitrary number of planes, and the common intersections be either circles, ellipses, parabolas or hyperbolas, but never another curve, then the surface in question will be a parabolic conoid or hyperbolic.*

[1] Translation available in Smith, *A Source Book in Mathematics*

[2] Let us recall that Archimedes called *conoid* the elliptical paraboloids of revolution and the hyperboloids of revolution (two-layered); and he called ellipsoids of revolution, *spheroids.*

For loci on conical surfaces:

5. *If a given surface is cut by an arbitrary number of planes, and the common intersection be always a straight line, a circle, an ellipse, a parabola or a hyperbola, but never another curve, then the surface in question will be a cone.*

For loci on cylindrical surfaces:

6. *If a given surface is cut by an arbitrary number of planes, and the common intersection be either a straight line, a circle or an ellipse, but never anything else, then the surface in question will be a cylinder.*

But loci often present themselves whose sections (cuts) are straight lines, parabolas and hyperbolas, and nothing else, as the analysis of the question will soon show. Therefore it is suitable, or perhaps even absolutely necessary for this study, to constitute *a new species of cylinders having as parallel bases either parabolas or hyperbolas, and straight lines for their edges, self-parallel and joining the bases,* by analogy with ordinary cylinders. It follows that no planar section of such a cylinder will be a circle or an ellipse. Just like ordinary cylinders, these new cylinders may moreover be right or oblique, as the analysis of the proposed locus will indicate.

I repeat that problems of loci necessarily lead to such cylinders; therefore their invention and definition must not be regarded as useless.

What is more, before going further, I will say that the constructions of Archimedes for spheroids and conoids do not suffice for our goal; indeed, problems force us to consider obliques and not only rights.

From what we have proposed, there first result very beautiful loci on spherical surfaces:

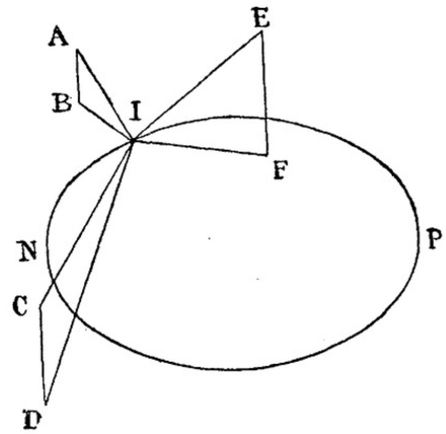
If an arbitrary number of points in an arbitrary number of planes be given, and we draw lines from them running through a single point, and the sum of the squares of the drawn lines be equal to a given area, then that single point will be on a spherical surface or on a sphere of given position.[3] Here we can, in fact, say a sphere, in imitation of Euclid and ancient geometers, who meant by *circle* the circumference and not the area of the circle; in any case, the point in question will lie on a surface of this nature.

Indeed, let us take a fixed arbitrary plane, and in it, following the rules given elsewhere for plane and solid loci, let us find the locus of all points for which the sum of the squares of the distances to given points be equal to a given area.

This is easy: let us suppose the problem to be solved, and let curve NIP be assumed as the locus in the plane we are considering (*fig.* 89) Let us drop onto the plane, from the points A, E, C given by hypothesis, the normals AB, EF, CF. The plane's position being given, these normals AB, EF, CD dropped from the known points A, E, C, will also be considered as known, as will their points of intersection with the plane B, F, D. Let us take on the locus NIP an arbitrary point I, and let us join AI, BI, EI, IF, CI, DI.

[3] That is, all the points that satisfy the condition collectively lie on a sphere. (Ed.)

Fig. 89.



The lines AI, EI, CI joining the given points A, C, E, with the point I on the locus, the sum of the squares of these lines will be equal to the given area. If we take the squares of the normals AB, EF, CD, which are known (as we have proved), the difference will be $BI^2 + FI^2 + DI^2$, which sum will be known. Yet the points B, F, D are known to be in the given plane, as we have seen; thus, we have lines BI, FI, DI drawn from points B, F, D given in the same plane, lines converging on the same point from locations in the same plane, and for whom the sum of the squares is equal to a given area. According to a theorem of Apollonius which we have restored long ago, we know that locus NIP is a determinate circle.

A completely similar analysis will give the same consequences for any other plane that you wish, since these planes, of indefinite number, will always give circles as loci; following lemma 2, the sought surface is therefore a sphere.

In fact, when we seek a surface locus satisfying a condition, nothing prevents our imagining that the sought surface is cut by our chosen plane. But here the section can only be a circle, for we have proved that a circle is the locus satisfying the same condition as the sought surface. Therefore it is necessary that the circle be situated upon said surface. It is therefore clear that in the proposed case, the surface locus is always cut by a plane in a circle, and consequently it is a sphere.

We demonstrate the same for the following loci:

If from an arbitrary number of points, given in one or several planes, lines be drawn converging in the same point, and if the sum of the squares of a portion of the lines has, to the sum of the squares of the other lines, a given ratio or a given difference, either greater or smaller than a given quantity or a given ratio,[4] *then the point of convergence will be on a sphere of fixed position.*

[4] That is to say, generally, if it be a linear function. It may be in a ratio, have an additive-subtractive difference, or both. (Second sentence of footnote added - Ed.)

Analagous techniques will bring to light an infinity of very beautiful properties of the spherical surface.

Let there be an arbitrary number of given planes; if from a single point we draw to these given planes, at fixed angles, lines, whose sum of squares be equal to a given area, then this point will lie on the surface of a given spheroid.

Let us perform the analysis by taking, following the indicated method, an arbitrary plane of known position. Let us seek on it, following the rules for plane and solid loci, such as we have earlier exposed in the plane, the locus of points for which the sum of squares of lines drawn to given planes at given angles is equal to a given area.

The construction presents itself immediately. The plane that we have taken is in effect fixed in position just as the other given planes are. The intersections of this chosen plane with the given planes will thus also be known. Lines drawn from given planes to an arbitrary point on the proposed plane will therefore easily be given analytic expression. If the sum of their squares be deteremined and it be caused to be equal to a given area, then analysis will give as locus, in the proposed plane, a circle or an ellipse. Analysis can also prove that in any other plane, the locus cannot be of another nature. Therefore, it is clear, according to lemma 3, that the sought locus, whose sections are only circles or ellipses, is a spheroid.

If the sum of the squares of a portion of the lines so drawn is to the sum of the others in a given ratio or in a given difference, or if it is larger or smaller than a given quantity [[107]], then the sought surface is either a spheroid, a conoid, a cone, or a cylinder, as will be determined by suitably conducted analysis.

For example, if we give the ratio, then we will in general have a conoid surface; but if the given planes intersect following lines converging on the same point, then the surface will be conical. If the intersections of the given planes are parallel, the surface will be cylindrical. Moreover, we may have arrive at either an ordinary cylinder, or one of ours.

Practice will immediately discover which it is; I limit myself to giving general and summary guidelines, in order that an excessive number of examples not prevent the clear grasping of my method.

I have reserved for last an example of a plane locus, which would perhaps have better been placed first:

Let the position of a number of arbitrary planes be given; if to these planes we draw from a single point, at given angles, lines whose sum is equal to a given line, the point will be on a plane.

Following the indicated method, let us cut the given planes by an arbitrary plane, and let us seek on it the locus satisfying the condition, following the given method for plane loci. It will be a straight line, as analysis shows, and it will be the same for all the other plane sections. It is therefore clear, according to lemma 1, that the sought surface is a plane.

If the sum of a determinate portion of lines thus drawn is to the sum of the others, in a given ratio or a given difference, or if it is larger or smaller than a given quantity or ratio, the point will also on a determinate plane surface.

Moreover, in the preceding problems, if the given planes had been parallel, the locus would also have been a plane surface, a remark it is hardly necessary to make.

As a *finale*, I will add a notable extension of the three- or four-line locus of Apollonius.

Let three arbitrary planes be given. If from a given point, lines be drawn to the planes at given angles, and the lines be made such that the product of two among them be to the square of the third in a given ratio, then the locus of the point will be either a plane, a sphere, a spheroid, a conoid, a conic surface, or a cylindrical surface (old or new), depending on the different orientations of the given planes.

It is the same for four planes, as is easily seen.

The various cases, the condition-limits for the givens, and the infinite number of local problems and theorems that we have omitted for brevity, the demonstration of the presented lemmas, and everything which could require a longer explanation, will be handily supplied by any careful and reflective geometer who has read this writing. From now on, this subject, which appeared so singularly arduous, will be made easy to understand.