# On the Form of Imaginary Roots of Equations

Joseph-Louis Lagrange

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It seems that Analysts have always considered as true the proposition that all imaginary roots of equations can be reduced to the form

$$A + B\sqrt{-1}$$

where $A$ and $B$ are real quantities. But it is only in recent times that this theorem has been demonstrated in a rigorous and general manner.

The first demonstration of this beautiful Theorem is that found in the *Memoirs* of this Academy for the year 1746, and is thanks to M. d'Alembert. This demonstration is quite ingenious and it seems to me that it leaves nothing to be desired in terms of exactness. But it is indirect, drawing upon the consideration of curves and infinite series, and it naturally leads one to believe that it is possible to arrive at the same end by a simpler analysis, founded solely upon the theory of equations. Indeed, since the imaginary radical $\sqrt{-1}$ can have the sign $+$ or $-$ indifferently, it is clear that if an arbitrary equation has a root represented by $A + B\sqrt{-1}$, then there will have to be at the same time another root, $A - B\sqrt{-1}$. Thus each imaginary factor $x - A - B\sqrt{-1}$ will always be accompanied by the corresponding factor $x - A + B\sqrt{-1}$, such that the product of these factors will be

$$x^2 - 2Ax + A^2 + B^2,$$

which is a real factor of the second degree.

It follows from this that any equation will be decomposable into real factors of the first or second degree. Now this proposition naturally appears to be demonstrable solely by the principles of the theory of equations, and clearly for this it will suffice to prove that any equation of degree higher than the second can always be divided into two other equations whose coefficients are real quantities. That is the goal that M. Euler set himself to in the wise studies he put forward in the *Memoirs* of 1749, on the imaginary roots of equations. There, he considers separately the case where the exponent of the equation is a power of 2, and the case where the exponent is a power of 2 multiplied by an arbitrary odd number. In the latter case he finds that any equation of degree $2^n m$ ($m$ being odd) can be divided by an equation of degree $2^n$ whose coefficient of the second term will be determined by an equation of odd degree, which will consequently always have a real root.[1] From this M. Euler first concludes that the coefficients of the other terms will also have real values, because he assumes that by successive eliminations of powers of the coefficients above the first, by the aid of different equations of condition[2] that will hold between the coefficients, that it will always be possible to determine the relevant coefficients[3] by rational functions of the coefficient of the second term. This reduction indeed appears to be possible in general. There are nevertheless particular cases where it will not hold, and for which the demonstration of M. Euler will be consequently insufficient. But this demonstration is above all deficient in regard to the first case, in which the proposed equation is assumed to have a degree which is a power of 2.

The solution of this case appears at first glance to be much more difficult. For when we try to divide an equation of degree $2^n$ by another equation of some lesser degree, we always arrive at equations of even degree for the determination of its coefficients; such that in order to be certain that one of the coefficients is real, it will be necessary that the equation on which they depend will have a negative final [constant] term. When we decompose an equation of the fourth degree, whose second term has disappeared,[4] into two others of the second degree by following Descartes' method, we find that the coefficients of the second terms of these divisors are given by an equation of sixth degree, whose last term is necessarily negative, since it is equal to a square with negative sign. This observation lead M. Euler to think that the same could hold in any equation whose degree is a power of 2, and whose second term is similar zero, whenever one seeks to decompose it into

[1]An equation of odd degree will always have a real root because imaginary roots come in pairs, leaving at least one real root by itself. – JAR

[2]*

[3]*les coefficients dont il s'agit*

[4]is equal to zero

two others of degree less than half.[5] M. Euler set himself to demonstrate, from the very nature of the roots of the equation which would determine the coefficients of the second terms of these divisors, that this equation would always have a square with negative sign as its last term; but it must be stated that his reasoning is not conclusive, as M. le chevalier de Foncenex[6] has already remarked in the first volume of the *Miscellanea Taurinensia,* and as we shall demonstrate in more detail in this Memoir.

This reason had indeed goaded[7] the gifted Geometer of whom we speak to take another path to arrive at an exact demonstration of this Theorem, and one could not doubt[8] that the one he has given in the cited volume has the advantage of elegance and simplicity. But, on the other hand, it is also subject to some of the difficulties which obtain against M. Euler, and which come from the false assumption that as soon as one of the divisors of an arbitrary equation is real, all the others must also be real.

It seems, therefore, from everything that we have just said, that our Theorem has not yet been demonstrated in as direct and rigorous a manner as would be desired. Since I have particularly applied myself to the perfection of the theory of equations, I believed that I must also attach myself to the discussion of such an important point of this theory: this is the object that I have proposed to myself in this Memoir. By adding that which is lacking in the demonstration of M. Euler, I will attempt to leave no difficulty or uncertainty in this matter.

1. It is well known that an equation of odd degree necessarily has a real positive root if its last term is negative, or a real negative root if its last term is positive. Furthermore, an equation of even degree necessarily has two real roots, one positive and the other negative, when its last term is negative.[9]

These Theorems are so well known, that we do not believe it necessary to take the time to prove them. It is true that the demonstration that is ordinarily given is unnatural,[10] being taken from the consideration of curved lines; but we have given a more direct demonstration, derived solely from the

[5] *moitié* – but that doesn't seem to make sense...
[6] Sir de Foncenex
[7] engagé
[8] *L'on ne saurait disconvenir... n'ait l'avantage de...*
[9] Need one read any further? He has assumed the existence of roots in his first sentence!
[10] *peu naturelle*

principles of the composition of equations.[11]

Beyond the preceding cases there has yet to be discovered a general characteristic by which one can tell *a priori* whether an equation does or does not have real roots. We give ourselves the task of giving our studies on this question, which can justly be considered as one of the most important in the theory of equations, on another occasion.

2. Having made these assumptions, it is clear first of all that any equation of odd degree of the form

$$x^{2m+1} - Ax^{2m} + Bx^{2m-1} - Cx^{2m-2} + ... - K = 0$$

can be reduced to a degree one lower, *i.e.,* to the even power just below it.

For, since we are certain that this equation must have a real root, then, if we denote this root by $a$, we will have

$$a^{2m+1} - Aa^{2m} + Ba^{2m-1} - Ca^{2m-2}+,,, -K = 0,$$

and therefore,

$$K = a^{2m+1} - Aa^{2m} + ba^{2m-1} - Ca^{2m-2} + ...,$$

which, upon being substituted into the preceding equation, yields:

$$\left(x^{2m+1} - a^{2m+1}\right) - A\left(x^{2m} - a^{2m}\right) + B\left(x^{2m-1} - a^{2m-1}\right) - C\left(x^{2m-2} - a^{2m-2}\right) + ... = 0,$$

which naturally decomposes into these two equations:

$$x - a = 0,$$

$$x^{2m} + (a-A)x^{2m-1} + (a^2 - Aa + B)x^{2m-2} + (a^3 - Aa^2 + Ba - C)x^{2m-3} + ... = 0.$$

Thus it will suffice to consider equations of even degree.

3. skipped for now

4. These are the most natural of the methods for decomposing an arbitrary equation into two others of lower degree. But, it is not necessary to perform

[11]See the *Memoirs* from the year 1767, available in volume II, p. 541 of the *Œuvres de Lagrange*

this decomposition; our goal requires simply that we show that it may be accomplished without coming upon imaginary quantities.

Now if we assume that in equations containing the indeterminates M, N, P, ... we eliminate all the indeterminates except for one of them, M for example, we will have a final equation in M of a higher degree when the number of these equations is greater. The question is then reduced to knowing: first, whether this equation will have at least one real root; second, if the values of the other indeterminates N, P, ... corresponding to this root will also be real.

5.