Means

A discrete arithmetic mean is

$$\frac{a_1 + a_2 + a_3 + a_4 + ...a_n}{n}.$$

A continous arithmetic mean is

$$\frac{\int_0^n f(x)dx}{n}.$$

A discrete geometric mean is:

$$\sqrt[n]{a_1 a_2 a_3 a_4 ... a_n}.$$

What is a continuous geometric mean? Well, if we take the logarithm of the product of a number of terms, that is the same as the sum of the individual logarithms.[1] This allows us to use an integral to add them up, giving:

$$e^{\frac{1}{n}\int_0^n \log f(x)dx}$$

A discrete harmonic mean is:

$$\frac{2ab}{a+b}$$

or

$$\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}$$

[1] Thanks, Peter!

or

$$\frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+...+\frac{1}{a_n}}{n}}.$$

We could also say, if μ is the harmonic mean, that:

$$\frac{1}{\mu}=\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+...+\frac{1}{a_n}\right).$$

A continuous harmonic mean would be:

$$\frac{1}{\mu}=\frac{1}{n}\int_0^n \frac{dx}{f(x)}.$$

Gauss says on page 18 of the elliptical ring paper that:

$$\frac{1}{\mu}=\int\frac{dT}{2\pi\sqrt{mm\cos T^2+nn\sin T^2}}$$

Considering $\frac{1}{n}=\frac{1}{2\pi}$, this expression looks to me to be the the harmonic mean of the central distances of the points on an ellipse, if m is the semi-major axis, n the semi-minor, and T the changing eccentric anomaly. I don't know the context of this in Gauss's paper, but he does say that μ is the arithmetic-geometric mean of m and n, so I don't know how much context is needed.

I find the question of means to be very exciting, because you have to consider along what you will integrate:[2] in other words, you could get the average speed of a car by averaging the speedometer readings – taking one every mile, but you would be incorrect. If you took the average of the *inverses* of the speedometer readings, and then took the inverse of that average, you *would* have the proper average speed of the car. That works because the inverses of the speeds are the times, which you can integrate over distance to get the total time, which would give you the average speed for the trip. If you take the average of the speedometer by taking samples every minute instead of every mile, then you would also get the correct mean.

– Jason

[2]This is what makes chapter 48 so difficult. Kepler takes precisely this harmonic mean, although he doesn't say so in the *New Astronomy*. Without using it, I didn't get the same results that he got.