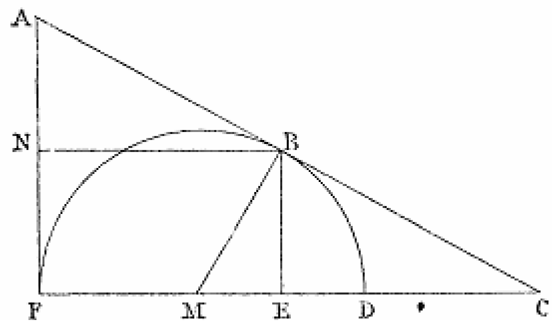
Fig. 100.



If, according to our method, we seek to construct the rectangle FE×EB by giving it a value, the question comes down to describing a hyperbola having AF and FC as its asymptotes and where the product of its abscissas FE and ordinates EB should have that given value; the points of intersection of the hyperbola and the semicircle will fulfill the question. But, since the product FE×EB must be a maximum, we must, in fact, make a hyperbola which has AF and FC as its asymptotes, and which, instead of intersecting the semicircle, is tangent to it instead, at B. For the points of contact determine maximum and minimum quantities.

Let us suppose the problem solved: if the hyperbola touches the semicircle at B, the tangent to the semicircle at B will also be tangent to the hyperbola. Let this line be ABC. It is tangent to the hyperbola at B and touches the asymptotes at A and C; therefore, according to Apollonius, AB = BC. Consequently, FE = EC and AF = 2BE = 2AN. But, since it is tangent to the circle, BA = AF; therefore BA = 2AN, and by the similarity of triangles, if M is the center, MB = 2ME. But the radius MB is given; therefore the point E will be known.

Similarly, we can in general reduce any search for a maximum or a minimum to the geometric construction of a tangent; but this does not in any way diminish the importance of the general method, since the construction of tangents depends on it, just as the determination of maxima and minima.

VI
On the Same Method

The theory of tangents is a result of the long-published method for the finding of maxima and minima, which permits the easy solution of all problems of limits, and notably the famous problems whose limit-conditions were considered difficult by Pappus (Book VII, preface).

The curved lines for which we are seeking tangents can be expressed by their specific properties, either by means of straight lines alone, or by means of a mixture of complex curves, as one wishes, with the help of straight lines or other curves.

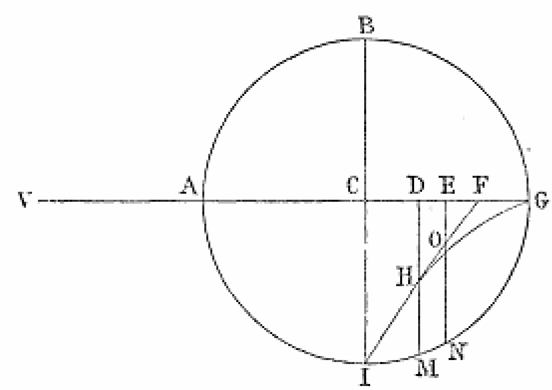
With our rule, we have already met the conditions for the first case which may have appeared to be difficult because it was too concise; however, it has nonetheless been recognized as legitimate.

In fact, in the plane of an arbitrary curve, we consider two lines given in position, one called the *diameter*, the other, the *ordinate*. We assume the tangent to be already found at a given point on the curve, and we consider by *ad-equality* the specific property of the curve, no longer on the curve itself, but on the sought-for tangent. Following our theory of maxima and minima, we eliminate those terms which ought to be eliminated, and arrive at an equality which determines the point of intersection of the tangent with the diameter, and then later the tangent itself.

To the numerous examples which I have already given, I will add that of the tangent to the *cissoid*, invented, it is said, by Diocles.

Let there be a circle wherein two diameters AG, BI (*fig.* 101) cut each other perpendicularly, and let there be cissoid IHG, on which, through any of its points, say H, we must draw the tangent.

Fig. 101.



Let us consider the problem to be solved, and suppose F to be the intersection of CG and the tangent HF. Let us call DF = a, and, in taking an arbitrary point E between D and F, let us say that DE = e.

Making use of the property specific to the cissoid that MD/DG = DG/DH, we will thus have to express analytically the *adequality* NE/EG ~ EG/EO, EO being the portion of the line EN between E and the tangent.

Let AD = z be given, DG = n be given, DH = r be given, and, as we have said, the unknown DF = a, the arbitrary DE = e.

We will have

$$EG = n - e, \qquad EO = (ra - re)/a, \qquad EN = \sqrt{(zn - ze + ne - e^2)}.$$

According to rule, we must consider the specific property, not on the curve, but on the tangent, and therefore state that NE/EG=EG/GO, EO being the ordinate of the tangent. Or, in analytical notation:

$$\sqrt{(zn - ze + ne - e^2)}/(n - e) \sim (n - e)/[(ra - re)/a].$$

Squaring, to remove the radical:

$$(zn - ze + ne - e^2)/(n^2 + e^2 - 2ne) \sim (n^2 + e^2 - 2ne)/[(r^2a^2 + r^2e^2 - 2r^2ae)/a^2].$$

Multiplying all the terms by a^2, and, according to rule, *adequalizing* the product of the extremes with the square of the mean, and removing the superfluous terms, pursuant to our method, we will finally have

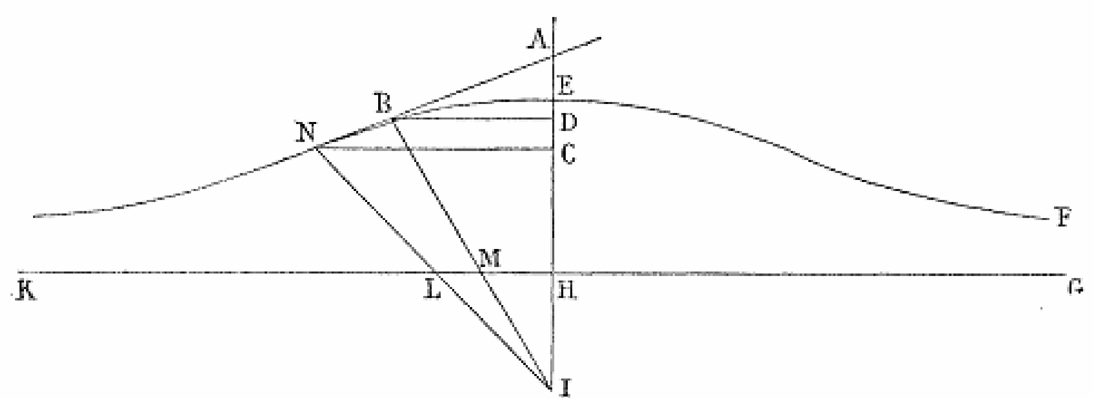
$$3za + na = 2zn.$$

Whence comes the following construction of the tangent: Extend radius CA to V such that AV = AC. Divide AD×DG by VD, make DF the quotient; connect FH. You will have the tangent of the cissoid.

We also indicate how to proceed for the *conchoid of Nicomedes*, but we will only sketch it out so as not to be too prolix.

Let the conchoid of Nicomedes be constructed in the figure as it is in Pappus and Eutocius (*fig.* 102). The pole is I, KG the asymptote, IHE the perpendicular to the asymptote, N a given point on the curve, through which we are to draw a tangent NBA meeting IE at A.

Fig. 102.



Let us suppose the problem solved, as above. Let us draw NC parallel to KG. By the property specific to the curve, we have LN = HE. Let us take an arbitrary point between C and E, say D, and draw DB through this point, parallel to CN, letting DB reach the tangent at B. Since the property specific to the curve must be considered on the tangent, let us join BI which meets KG at M. Following the rules of the art, we must ad-equate MB and HE. We will thus arrive at the sought-for equation. For this, we will say, as above, that CA = a, CD = e, EH = z, and we will call the other ordinates by their names. We will easily find the analytical expression of the line MB. We will then *adequalize* it, as has been said, to line HE, and solve the problem.

What I have said seems to suffice for the first case. It is true that there are an infinite number of artifices to shorten calculations in practice; but one can easily deduce them from what has come before.

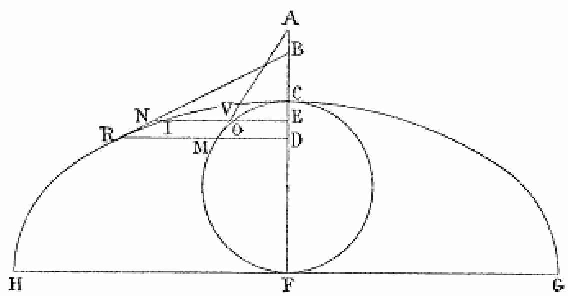
As for the second case that M. Descartes considered difficult, and for whom nothing is, we have brought it to satisfaction by way of a very elegant and subtle method.

As long as the terms are formed only by straight lines, we may find and designate them according to the preceding rule. Moreover, to avoid radicals, we may substitute, in place of the ordinates to the curve, the ordinates of the tangents found by the preceding method. Finally, and most importantly, we may substitute, for the arcs of the curves, the corresponding lengths of the already-found tangents, and arrive at the *adequality*, as we have indicated: thus we will easily satisfy the question.

Let us take the curve of M. de Roberval – the cycloid – as an example.

Let HBIC be the curve (*fig.* 103), C its summit, and CF the axis; let us describe the semicircle COMF, and take an arbitrary point on the curve, say R, from which the tangent RB is to be drawn.

Fig. 103.



Let us draw from this point R, perpendicularly to CDF, the line RMD, cutting the semicircle at M. The property specific to this curve is that the line RD is equal to the sum of the arc of the circle CM and the ordinate DM. Let us then draw, following our preceding method, MA, the tangent to the circle (the same process would indeed be applicable if the curve COM were of another nature). Let us suppose the construction performed, and let the unknown DB = a, the lines found by construction: DA = b, MA = d; the givens MD = r, RD = z, the given arc of the circle CM = n, and the arbitrary line DE = e.

From E draw EOVIN parallel to the line RMD; we have $a/(a - e) = z/\text{NIVOE}$, whence we have NIVOE = $(za - ze)/a$.

Therefore it is necessary to *ad-equate* (because of the specific property of the curve which is to be considered on the tangent) this line $(za - ze)/a$ to the sum OE + arcCO.

But arcCO = arcCM – arcMO. Therefore $(za\text{-}ze)/a \sim$ OE + arcCM – arcMO.

To obtain the analytic expression of the three last terms, while avoiding radicals, we can, according to the preceding remark, substitute the ordinate of the tangent EV for OE, and the portion of the tangent MV for arc MO.

To find the analytic expression of EV, we moreover have $b/(b - e)=r/\text{EV}$, whence EV = $(rb - re)/b$.

For MV, by reason of similar triangles, we have, as above, $b/d = e/\text{MV}$, whence MV = de/b.

Finally we have arcCM = n. Thus, we have, analytically:

$$(za\text{-}ze)/a \sim (rb - re)/b + \text{n} - de/b.$$

Multiplying both sides by ab:

$$zba - zbe \sim rba - rae + bna - dae.$$

But, from the property of the curve, we know that $z = r + n$, and therefore $zba = rba + bna$.

Removing the common terms, we have

$$zbe \sim rae + dae.$$

Let us divide by e; since there are no more superfluous terms here, there are no other eliminations to make:

$$zb = ra + da, \qquad \text{whence} \qquad (r + d)/b = z/a.$$

For the construction, we will then make (MA + MD)/DA = RD/DB; we will join BR which will touch curve CR.

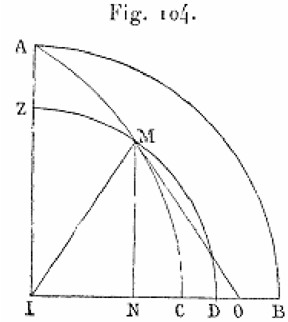
But since (MA+MD)/DA=MD/DC, as is easy to demonstrate, we can have MD/DC = RD/DB, or, to make the construction more elegant, we may join MC and then draw RB parallel.

The same method will give the tangents to all curves of this type. We have indicated their general construction a long time ago.

Since it has been proposed to find the *tangent of the quadratrix of Dinostratus*, here is how we can construct it according to the preceding method.

Let AIB be a quarter of a circle (*fig.* 104), AMC the quadratrix, from which we must draw the tangent at a given point M. I join MI, and then with I as the center and IM the radius, I draw the quarter circle ZMD. Drawing the perpendicular MN, I make MN/IM = arcMD/IO. I join MO which will be tangent to the quadratrix; this should be sufficient.

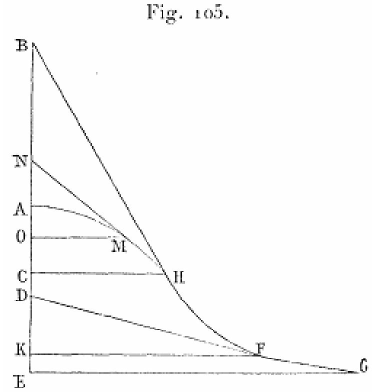
Fig. 104.



However, it often occurs that the curvature changes, as for the conchoids of Nicodemus (1st case) and for all species except for the first one, the curve of M. de Roberval (2nd case). To be able to draw the curve well, it is suitable therefore to mathematically research the points of inflection, where the curvature changes from convex to concave, or the inverse. This question can be elegantly resolved by the method of *maximum and minimum*, thanks to the following general lemma:

Let there be, for example, the curve AHFG (*fig.* 105) *whose curvature changes at point* H. *Draw the tangent* HB *and the ordinate* HC; *the angle* HBC *will be the minimum among all those angles which the tangent makes with axis* ACD *when it be below or above point* H, as is easy to demonstrate.

Fig. 105.



Let us take point M above point H. The tangent to this point will reach the axis at a point between A and B – let it be N. The angle at N will therefore be greater than the angle at B.

Similarly, if we take the point F below H, then the point D where the tangent DF meets the axis will be below B. Moreover, the tangent DF will meet the tangent BH on the side of FH. The angle at D will therefore be greater than the angle at B.

We will not pursue all cases, preferring only to indicate the mode of study, since the forms of curves vary infinitely.

Therefore to find, for example, the point H on the shape, we will first seek, following the preceding method, the property of the tangent at an arbitrary point of the curve. Since, by the doctrine *of maxima and minima*, we will determine the point H such that by drawing the perpendicular HC and the tangent HB, the ratio HC/CB will be a minimum. For thus the angle at B will be a minimum. I say that the point H thus determined will be that where the change in curvature is found.

The same method *of maxima and minima* gives also, by a singular expedient, the determination of the center of gravity, as I have indicated to M. de Roberval in the past.

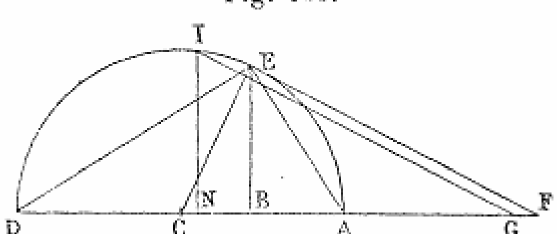
But, as a crowning achievement, we can even *find the asymptotes of a given curve*, a study which leads to remarkable properties for indefinite curves. We shall develop and demonstrate them more at length at some future time.

VII
Problem Sent to the Rev. Father Mersenne
on the 10th of November, 1642

Find the cylinder of maximum surface area inscribed in a given sphere.

Let there be given a sphere of diameter AD (*fig.* 106), with center C. It is demanded to inscribe within it the cylinder of maximum surface.

Fig. 106.



Let us suppose the problem solved. Let DE be the diameter of the base of the cylinder, EA its side (we can indeed give this position on the cylinder, the angle inscribed in the semicircle being right). The surface of the cylinder is proportional to $DE^2 + 2DE \cdot EA$; it is therefore necessary to find the maximum of the sum $DE^2 + 2DE \cdot EA$.

If the perpendicular EB be dropped, we have for one term $DE^2 = AD \cdot DB$, and for the other $DE \cdot EA = AD \cdot BE$. Thus, we must find the maximum of the sum $AD \cdot DB + 2AD \cdot BE$, or, by dividing the terms by the given line AD, the maxi
mum of the sum $DB + 2BE$.