Contribution to the Theory

of

Algebraic Equations

by

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This essay will treat of two subjects concerning algebraic equations. In the first place, I will present my 50 year-old proof on the fundamental theorem of algebra, in a changed form and with weighty additions. The second part is dedicated to a special handling of algebraic equations with three terms, and contains methods to determine not only the real, but also the imaginary roots of such equations.

First Section

The essay presented in 1799, *Demonstratio nova theorematis, omnem functionem algebraicum rationalam integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*, had a double purpose, namely, first to indicate the insufficiency and illusory nature of the theretofore attempted proofs of this important theorem of the theory of algebraic equations, and to give a new complete, strong proof. It is unimportant to return to the first objective. I have added two more proofs to this then-newly given proof, and a fourth is ((primarily for Cauchy presented.))

These four proofs rest upon that many [four] different foundations, but they all agree in primarily proving only the presence of simple factors of the related functions. This does not diminish the strength of the proof since it is clear, that when a factor is removed from the given function, a similar function of lower order remains to which the theorem can again be applied, which through repeated application will finally bring about the resolution of the original function into factors of the indicated type. From this, the argument, when proven to be appropriate, easily demonstrates more completely the direct presence of all the factors. That the first proof is of this sort [demonstrating the presence of all the factors], I have already indicated in the aforesaid [1799] essay (art. 23), without further drawing it out: this shall now become completed, and I will now use this opportunity to repeat the main point of the whole proof in a transformed, and (I believe) clearer form. The

1799 proof concerned itself with the outer clothing of the theorem, that the function $x^n + \mathrm{A}x^{n-1} + \mathrm{B}x^{n-2} +$ etc. can be dissected [resolved] into real factors of the first or second degree, and any interference by imaginary magnitudes was avoided for this [aim]. Presently, since everyone is well versed in the concept of complex magnitudes, it appears appropriate to express every form ←** with them and hence to express the theorem: that every function can be resolved into *n simple* factors, where the constant parts of these factors use not only real magnitudes, but admit also of complex magnitudes. By this clothing the theorem obtains generality, because then the limiting of the coefficients *A, B,* etc. is not enforced, but to each of these values the same permission is granted [to be complex].

1.

We conceive, accordingly, the function of the [undetermined] magnitude x

$$x^n + A\,x^{n-1} + B\,x^{n-2} + \text{etc.} + M\,x + N = X$$

where *A, B, ... M, N* represent determined real or imaginary coefficients. The connection between the roots of the equation X = 0 and the simple factors of X is known from elementary algebra. Namely, it happens that when that equation is satisfied through the substitution $x = p$, then $x - p$ is a factor of X, and that be there n different ways to satisfy that equation, namely through $x = p,\ x = p',\ x = p''$, etc., then the product $(x - p)\,(x - p')\,(x - p'')\ldots$ will be identical to X. Under certain circumstances, however, a solution, such as $x = p$, can also give rise, in X, to the factor $(x - p)^2$, or $(x - p)^3$ or any higher power, in which case one conceives the root p as doubly present, triply present, etc.

One requires only the proof that the function X contains *one* certain simple factor, for which it is sufficient to establish the presence of any root of the equation X = 0. Should however the complete factorability of the function in simple factors be directly proved, then it is indicated that the satisfying of the equation X = 0 will be accomplished, either through n unequal values of x, or through a definite smaller number of unlike solutions, of which, however, a part bears on itself the character of multiple roots, in such a way, that the sum of all the unequal and equal roots becomes = n.